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A queueing system with n phases of service, vacations and retrial customers

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Abstract

A queueing system with a single server providing n phases of service in succession is considered. Every customer receives service in all phases. When a customer completes his service in the i^{th} phase he decides either to proceed to the next phase of service or to join the K_i retrial box ($i = 1, 2, ..., n-1$), from where he repeats the demand for the $(i+1)^{th}$ phase of service after a random amount of time and independently to the other customers in the system. When there are no more customers waiting in the ordinary queue (first stage), the server departs for a single vacation of an arbitrarily distributed lenght. The arrival process is assumed to be Poisson and all service times are arbitrarily distributed. For such a system, the mean number of customers in the ordinary queue and in each retrial box separately are obtained, and used to investigate numerically system performance.

Keywords: Poisson arrivals, n-phase service, retrial queues, general services, single vacation.

1 Introduction

Queueing systems in which the server provides to each customer a number of phases of heterogeneous service in succession, can be proved very useful to model computer networks, production lines and telecommunication systems where messages are processed in n stages by a single server.

Such kind of systems, with only two phases of service, have firstly discussed by Krishna and Lee [9] and Doshi [5], while more recently in a series of works (Madan [13], Choi and Kim [2], Choudhury and Madan [4], Katayama and Kobayashi [7], Ke [8]), the previous results, are extended to include systems allowing server vacations, Bernoulli feedback, N-policy, exhaustive or gated bulk service, startup times etc., but again for models with only two phases of service. Moreover in all papers mentioned above one can find important applications to computer communication, production and manufacturing systems, central processor units and multimedia communications.

Kumar, Vijayakumar and Arivudainambi [11] and Choudhury [3] are the first who imposed the concept of "retrial customers " in the two phase service models. Retrial queueing systems are characterized by the fact that an arriving customer who finds the server unavailable does not wait in a queue but instead he leaves the system, joining the so called retrial box, from where he repeats the demand for service later. Practical use of retrial queueing systems arises in telephone-switching systems and in telecommunication and computer networks. For complete surveys of past papers on such kind of models see Falin and Templeton [6] , Kulkarni and Liang [10] and Artalejo [1]. Kumar et.al. [11] considered a two phase service system where an arriving customer who finds the server unavailable joins the retrial box from where only the first customer can retry for service after an arbitrarily distributed time period while in the work of Choudhury [3] the investigated two phase model includes Bernoulli server vacations and linear retrial policy. We have to observe here that in both papers the service procedure contains only two phases of service and also there is not any ordinary queue and all "waiting " customers are placed in the retrial box.

In the work here we consider, for first time in the literature, a model with n phases of service and $n-1$ retrial boxes, $K_1, K_2, ..., K_{n-1}$ say. All arriving customers are placed, upon arrival, in an ordinary queue (first stage) to receive service. When a customer completes his first phase service then, with probability $1-p_1$, he proceeds to the second phase while, with probability p_1, he leaves the system and joins the K_1 retrial box. This procedure is repeated in each stage and so, when a customer completes his i^{th} phase service, then either, with probability $1-p_i$, he proceeds to the $(i+1)^{th}$ phase, or with probability p_i he joins the K_i retrial box. The customers in each retrial box retry, after a random amount of time and independently to each other, to find the server available and to proceed to the next phase of their service. Note here that every customer can join more than one retrial boxes during his service procedure. Moreover, when there are no more customers for service in the ordinary queue, the server departs for a single vacation (update devices, maintenance, etc.) of arbitrarily distributed length. We have to point out here that in our model, and at any time, an ordinary and $n-1$ retrial queues must be taken in to account and so the way to handle the situation becomes much more complicated.

Our system can be used to model any situation with many stages of service, where in each stage a control and a separation of the serviced units must be taken place, and if a unit satisfies some quality standards then it proceeds immediately to the next phase of service, while, if the quality of the unit is poor, then it is removed from the system and repeats its attempt to continue its service procedure later when the server is free from high quality units. As one understand a such kind of situation arise often in packet transmissions, in manufacturing systems, in central processors, in multimedia communications etc..

The article is organized as follows. A full description of the model is given in section 2. Some, very useful for the analysis results, on the customer total service cycle and server busy and vacation periods, are given in section 3, while a system states analysis is performed in section 4. In section 5 the mean number of customers in the ordinary queue and in each retrial box separately are obtained, and used to produce, in section 6, numerical results and to compare numerically system performance under various changes of the parameters.

2 The model

Consider a queueing system consisting of n phases of service and a single server who follows the customer in service when he passes from one phase to the next. Customers arrive to the system according to a *Poisson* distribution parameter λ, and are placed in a single queue (first phase) waiting to be served. When a customer finishes his service in the i^{th} phase $(i = 1, 2, ..., n)$ then either he goes to the $(i+1)^{th}$ phase with probability $1-p_i$, or he departs from the system with probability p_i and joins the K_i^{th} retrial box $(i = 1, 2, ..., n-1)$ from where he retries, independently to the other customers in the box, after an exponential time parameter μ_i, to find the server idle and to proceed to the $(i+1)^{th}$ phase of service. In case the customer chooses to depart and to join the retrial box, the server starts immediately to serve in the first phase the next customer in queue (if any). Every time the server becomes idle (no customers waiting in the ordinary queue) he departs for a single vacation U_0 which length is arbitrarily distributed with distribution function (D.F.) $B_0(x)$, probability density function (p.d.f.) $b_0(x)$ and finite mean value $\bar{b}_0$ and second moment about zero $\bar{b}_0^{(2)}$. If the server, upon returning from a vacation, finds customers waiting for service in the first stage (ordinary queue) he starts serving them immediately, while if there are no customers waiting, he remains idle awaiting the first arrival, from outside or from a retrial box, to start the service procedure again.

Let us call P_1 customers the ordinary customers who are queued up and wait to be served and P_i customers $(i = 2, 3, ..., n)$ those who joined the K_{i-1}^{th} retrial box. Note here that any customer can join, during his service procedure, a number of retrial boxes and so a P_i customer is called P_i customer as far as he continues his service procedure passing from one phase of service to the next without joining another retrial box, while if a P_i customer joins in the sequel the K_{j-1}^{th} retrial box $(j > i)$ then he becomes a P_j customer. The service time of a P_i customer in the j^{th} phase, B_{ij} say, is assumed to be arbitrarily distributed with D.F. $B_{ij}(x)$, p.d.f. $b_{ij}(x)$ and finite mean value $\bar{b}_{ij}$ and second moment about zero $\bar{b}_{ij}^{(2)}$ ($B_{ij}(x)$, $b_{ij}(x)$, $\bar{b}_{ij}, \bar{b}_{ij}^{(2)}$ do not exist of course for $j < i$). Finally all random variables defined above are assumed to be independent.

3 General Results

If a customer does not join any retrial box during his service procedure (with probability $\bar{p}_0 = \prod_{i=1}^{n-1}(1-p_i)$) then his **total service cycle** will be $\bar{R}_0 =$

$\sum_{j=1}^{n} B_{1j}$ with LST of its p.d.f.

$$\bar{r}_0(s) = \prod_{j=1}^{n} \beta_{1j}^*(s),$$

with $\beta_{ij}^*(\cdot)$ the LST of $b_{ij}(\cdot)$. Let us suppose now that an arriving customer joins r retrial boxes ($r = 1, 2, ..., n-1$) during his service procedure, for example he joins the retrial boxes K_{m_1}, K_{m_2}, ..., K_{m_r}. Then it is clear that $m_1 = 1, 2, ..., n-r$, $m_2 = m_1+1, ..., n-r+1$, and so on, until $m_r = m_{r-1}+1, ..., n-1$. Moreover the probability of this event is

$$\bar{p}_{m_1 m_2 \dots m_r} = p_{m_1} p_{m_2} \cdots p_{m_r} \prod_{\substack{i=1 \\ i \neq m_1, \dots, m_r}}^{n-1} (1 - p_i),$$

while the duration of the customer's **total service cycle** in this case is (with $m_0 \equiv 0$)

$$\bar{R}_{m_1 m_2 \dots m_r} = \sum_{i=1}^{r} (\sum_{j=m_{i-1}+1}^{m_i} B_{m_{i-1}+1j} + \overline{V}) + \sum_{j=m_r+1}^{n} B_{m_r+1j},$$

and the LST of its p.d.f.

$$\bar{r}_{m_1 m_2 \dots m_r}(s) = (\overline{v}^*(s))^r \prod_{i=1}^{r+1} \prod_{j=m_{i-1}+1}^{m_i} \beta_{m_{i-1}+1j}^*(s),$$

with $m_{r+1} \equiv n$. Note here that $\overline{V}$ is the delay incurrent due to server absence (in vacations) that precedes the service of every customer emerging from a retrial box. This absence can be either of a single duration U_0, if no customers arrive from outside during the vacation U_0, or of a multiple duration, if at least one customer arrives during U_0, in which case the server has to repeat the vacation as soon as he finishes the busy period of P_1 customers and before he becomes available to the customer emerging from the retrial box. Thus the p.d.f. $\overline{v}(t)$ of $\overline{V}$ satisfies

$$\overline{v}(t) = e^{-\lambda t} b_0(t) + (1 - e^{-\lambda t}) b_0(t) * \overline{v}(t),$$

with LST

$$\overline{v}^*(s) = \frac{\beta_0^*(\lambda + s)}{1 + \beta_0^*(\lambda + s) - \beta_0^*(s)}.$$

Thus the LST of the p.d.f. of the customer's **total service cycle** $\bar{R}$ is given by

$$\bar{r}(s) = \bar{p}_0 \bar{r}_0(s) + \sum_{r=1}^{n-1} \sum_{m_1=1}^{n-r} \sum_{m_2=m_1+1}^{n-r+1} \cdots \sum_{m_r=m_{r-1}+1}^{n-1} \bar{p}_{m_1 m_2 \dots m_r} \bar{r}_{m_1 m_2 \dots m_r}(s),$$

and if we take derivatives above, at $s=0$, we arrive after some algebra at

$$\rho^* \equiv -\lambda \frac{d}{ds}\bar{r}(s)|_{s=0} = \lambda E(\bar{R}) = \sum_{j=1}^{n}(\rho_j + \rho_{0j}), \tag{1}$$

where

$$\begin{aligned} \rho_{0j} &= \tfrac{\lambda \bar{b}_0}{\beta_0^*(\lambda)} p_{j-1}, \qquad j = 2, 3, ..., n, \\ \rho_j &= \lambda p_{j-1}[\bar{b}_{jj} + \textstyle\sum_{k=j+1}^{n} \bar{b}_{jk} \prod_{m=j}^{k-1}(1-p_m)], \qquad j = 1, 2, ..., n. \end{aligned} \tag{2}$$

with $p_0 \equiv 1$, $\rho_{01} \equiv 0$. Thus ρ^* must be considered as the mean number of new customers arriving during $\bar{B}$ and so, for an ergodic system, we have to assume $\rho^* < 1$.

Let now S_j be the time interval from the epoch at which a P_j customer starts his service in the j^{th} phase until the epoch he either completes his service procedure and depart from the system or he joins another retrial box and releases the server. Let also $N_i(S_j)$ be the new P_i customers during S_j. Note here that during S_j we can have only new P_1customers (external arrivals) and/or one and only one new P_{j+1} or P_{j+2} or ...or P_n customer according to the retrial box that this specific P_j customer will join next. Define finally

$$a^{(j)}(t, k_1,\ k_{j+1}, ..., k_n)dt = P[N_i(S_j) = k_i \ \ i = 1, j+1, ..., n,\ \ t < S_j \le t + dt],$$
$$a_j(z_1, z_{j+1}, ..., z_n) = $$
$$\sum_{k_1=0}^{\infty} \sum_{k_{j+1}=0}^{1} \cdots \sum_{k_n=0}^{1} z_1^{k_1} z_{j+1}^{k_{j+1}} ... z_n^{k_n} \int_{t=0}^{\infty} a^{(j)}(t, k_1,\ k_{j+1}, ..., k_n)dt.$$

Then it is easy to understand that, for any $j = 1, 2, ..., n$,

$$a_j(z_1, z_{j+1}, ..., z_n) = \sum_{m=j}^{n} p_m z_{m+1} \prod_{l=j}^{m-1}(1-p_l) \prod_{k=j}^{m} \beta_{jk}^*(\lambda - \lambda z_1), \tag{3}$$

with $p_n \equiv 1,\ z_{n+1} \equiv 1$. Moreover from relation (2)

$$\frac{d}{dz_1} a_1(z_1, 1, 1, ..., 1)|_{z_1=1} = \rho_1.$$

In general and for any $x_j \ \ j = 1, 2, ..., n$, let as denote, for simplicity, by $\boldsymbol{x}$ the $(1 \times n-1)$ vector $\boldsymbol{x} = (x_2, x_3, ..., x_n)$ and by $\bar{\boldsymbol{x}}$ the $(1 \times n)$ vector $\bar{\boldsymbol{x}} = (x_1, x_2, ..., x_n)$. To proceed further we need the following Lemma the proof of which is a simple application of the well known theorem of Takacs [15].

Lemma 1 *If (i) $|z_k| < 1$ for any specific $k = 2, ..., n$, and $|z_m| \le 1$ for all other $2 \le m \le n$ with $m \ne k$, or (ii) $|z_m| \le 1$, for all $2 \le m \le n$ and $\rho_1 > 1$, then the relation*

$$z_1 - a_1(z_1, z_2, ..., z_n), \tag{4}$$

has one and only one zero, $z_1 = x(\boldsymbol{z})$ say, inside the region $|z_1| < 1$. Specifically for $\boldsymbol{z} = \mathbf{1}$, $x(\mathbf{1})$ is the smallest positive real root of (4) with $x(\mathbf{1}) < 1$ if $\rho_1 > 1$ and $x(\mathbf{1}) = 1$ for $\rho_1 \le 1$.

Let now $T^{(i)}$ be the duration of a busy period of P_1 customers starting with i P_1 customers and $N_j(T^{(i)})$ be the number of new P_j customers (joining the K_{j-1}^{th} retrial box) during $T^{(i)}$. Define

$$g_{\,k}^{(i)}(t)dt = P[N_j(T^{(i)}) = k_j \;\; j = 2,3,...,n, \;\; t < T^{(i)} \leq t + dt],$$
$$G^{(i)}(s,z) = \sum_{k=0}^{\infty} z^k \int_{t=0}^{\infty} e^{-st} g_{\,k}^{(i)}(t)dt.$$

Now it is easy to see (Theorem 1 in Langaris and Katsaros [12]) that

$$G^{(i)}(0,z) = x^i(z),$$

where $x(z)$ the only zero of $z_1 - a_1(z_1, z_2, ..., z_n)$ in $|z_1| < 1$.

Let now V be the time interval from the epoch the server departs for a single vacation until the epoch he becomes idle for the first time. Denote also $N_j(V)$ the number of new P_j customers during V. If we define

$$v_k(t)dt = P[N_j(V) = k_j \;\; j = 2,3,...,n, \;\; t < V \leq t + dt],$$
$$v^*(s,z) = \sum_{k=0}^{\infty} z^k \int_{t=0}^{\infty} e^{-st} v_k(t)dt,$$

then

$$v_k(t) = e^{-\lambda t} b_0(t) \delta_{\{k=0\}} + \sum_{m=0}^{k} \sum_{i=1}^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t} b_0(t) * g_{\,m}^{(i)}(t) * v_{k-m}(t),$$

where

$$\delta_{\{A\}} = \begin{cases} 1 & if \;\; A \;\; holds \\ 0 & otherwise, \end{cases}$$

and so

$$v^*(0,z) = \frac{\beta_0(\lambda)}{1 + \beta_0(\lambda) - \beta_0(\lambda - \lambda x(z))}. \quad (5)$$

Let $D^{(i)}$ the time interval from the epoch a P_i $i = 2,3,...,n$ retrial customer finds a position for service until the epoch that the server departs for a vacation, and denote by $N_j(D^{(i)})$ the number of the new P_j customers during $D^{(i)}$. Define finally

$$d_{\,k}^{(i)}(t)dt = P[N_j(D^{(i)}) = k_j \;\; j = 2,3,...,n, \;\; t < D^{(i)} \leq t + dt],$$
$$D^{(i)}(s,z) = \sum_{k=0}^{\infty} z^k \int_{t=0}^{\infty} e^{-st} d_{\,k}^{(i)}(t)dt,$$

then

$$d_{\,k}^{(i)}(t) = e^{-\lambda t} \sum_{r=i}^{n} s_{ir}(t) + \sum_{m=1}^{\infty} \frac{(\lambda t)^m}{m!} e^{-\lambda t} \sum_{r=i}^{n} s_{ir}(t) * g_{k-1_{r+1}}^{(m)}(t), \quad (6)$$

where $\mathbf{1}_j = (0, ..., 0, 1, 0, ..., 0)$ with the 1 in the j^{th} position and

$$s_{ir}(t) = (1-p_i)b_{ii}(t) * ... * (1-p_{r-1})b_{ir-1}(t) * p_r b_{ir}(t), \qquad r = i, i+1, ..., n,$$

is in fact the total time the P_i retrial customer holds the server from the epoch he finds a position for service until the epoch he joins the K_r^{th} retrial box and becomes a P_{r+1} customer or departs from the system (case $r = n$). By taking LST in (6) above we arrive at

$$D^{(i)}(s, z) = a_i(x(z), z_{i+1}, ..., z_n),$$

where the function $a_i(x(z), z_{i+1}, ..., z_n)$ has been defined in (3).

Define finally $C^{(i)}$ as the time interval from the epoch at which a P_i customer finds a position for service until the epoch the server becomes for the first time idle and ready to accept the next customer from outside or from a retrial box. If $N_j(C^{(i)})$ is the number of new P_j customers during $C^{(i)}$ and define

$$c^{(i)}_{k}(t)dt = P[N_j(C^{(i)}) = k_j \;\; j = 2, 3, ..., n, \;\; t < C^{(i)} \leq t + dt],$$
$$C^{(i)}(s, z) = \sum_{k=0}^{\infty} z^k \int_{t=0}^{\infty} e^{-st} c^{(i)}_{k}(t)dt,$$

then it is easy to realize that

$$C^{(i)}(0, z) = a_i(x(z), z_{i+1}, ..., z_n)v^*(0, z). \qquad (7)$$

We have to state here the following theorem. The proof is similar to the proof of Theorem 3.2 in Moutzoukis and Langaris [14] and it is omitted here.

Theorem 2 *For any permutation $(i_2, i_3, ..., i_n)$ of the set $(2, 3, ..., n)$ and for (a) $|z_{i_m}| < 1$ for any specific $m = j+1, ..., n$, and $|z_{i_r}| \leq 1$, for all other $r = j+1, ..., n$ with $r \neq m$, or (b) $|z_{i_r}| \leq 1$, for all $r = j+1, ..., n$, and $\bar{\rho}_{i_{j-1}} > 1$, or (c) $|z_{i_r}| \leq 1$, for all $r = j+1, ..., n$, and $\bar{\rho}_{i_j} > 1 \geq \bar{\rho}_{i_{j-1}}$, the equation*

$$z_{i_j} - C^{(i_j)}(0, \; w_{i_{j-1}}(z_{i_j}, z_{i_{j+1}}, ..., z_{i_n})) = 0, \qquad (8)$$

has, for $j = 2, 3, ..., n$, one and only one root, $z_{i_j} = x_{i_j}(z_{i_{j+1}}, ..., z_{i_n})$, $j \neq n$, $z_{i_n} = x_{i_n}$ say, inside the region $|z_{i_j}| < 1$, where the vector $w_{i_j}(z_{i_{j+1}}, z_{i_{j+2}}, ..., z_{i_n})$ is defined by

$$w_{i_1}(z_{i_2}, z_{i_3}, ..., z_{i_n}) = (z_2, z_3, ..., z_n),$$
$$w_{i_2}(z_{i_3}, z_{i_4}, ..., z_{i_n}) = w_{i_1}(x_{i_2}(z_{i_3}, ..., z_{i_n}), z_{i_3}, ..., z_{i_n}),$$
$$w_{i_k}(z_{i_{k+1}}, z_{i_{k+2}}, ..., z_{i_n}) = w_{i_{k-1}}(x_{i_k}(z_{i_{k+1}}, ..., z_{i_n}), z_{i_{k+1}}, ..., z_{i_n}), \quad k = 2, ..., n-1,$$

while $\bar{\rho}_{i_1} \equiv \rho_1$ and

$$\bar{\rho}_{i_j} = \frac{\partial}{\partial z_{i_j}} C^{(i_j)}(0, \; w_{i_{j-1}}(z_{i_j}, z_{i_{j+1}}, ..., z_{i_n}))|_{z_{i_j} = z_{i_{j+1}} = ... = z_{i_n} = 1}.$$

Moreover for real $z_{i_r} = 1$, $r = j+1, ..., n$, and $\bar{\rho}_{i_{j-1}} \leq 1$ the root $x_{i_j}(1, ..., 1)$ is the smallest positive real root of (8) with $x_{i_j}(1, ..., 1) < 1$ if $\bar{\rho}_{i_j} > 1$ and $x_{i_j}(1, ..., 1) = 1$ for $\bar{\rho}_{i_j} \leq 1$.

One can show here that, for any permutatation $(i_2, i_3, ..., i_n)$ of the set $(2, 3, ..., n)$, the last term $\bar{\rho}_{i_n}$ $(> \bar{\rho}_{i_{n-1}} > ... > \bar{\rho}_{i_2})$ is given by,

$$\bar{\rho}_{i_n} = \frac{\rho_{i_n} + \rho_{0i_n} + \delta_{\{i_n < n\}} p_{i_n - 1} \sum_{k=i_n+1}^{n} (\rho_k + \rho_{0k})}{1 - \rho_1 - \sum_{k=2}^{i_n - 1} (\rho_k + \rho_{0k}) - \delta_{\{i_n < n\}} (1 - p_{i_n - 1}) \sum_{k=i_n+1}^{n} (\rho_k + \rho_{0k})}, \tag{9}$$

and so it is clear comparing relations (1), (9) that, for $\rho^* < 1$, $\bar{\rho}_{i_n}$ (and all other $\bar{\rho}_{i_j}$) is always less than one.

4 Steady states analysis

Let us assume that a state of statistical equilibrium exists and let N_i, $i = 1, 2, ..., n$ denote the number of P_i customers in the system. Let also

$$\xi = \begin{cases} 0 & if\ server\ on\ vacation, \\ (i,j) & if\ server\ busy\ on\ j\ phase\ with\ P_i\ customer, \\ id & if\ server\ idle, \end{cases}$$

and

$$\begin{aligned} & q(\boldsymbol{k}) = P(\xi = id,\ N_1 = 0,\ N_m = k_m,\ m = 2, 3, ..., n), \\ & p_0(\bar{\boldsymbol{k}}, x)dx = P(\xi = 0,\ N_m = k_m,\ m = 1, 2, ..., n,\ \ x < \bar{U}_0(t) \le x + dx), \\ & p_{ij}(\bar{\boldsymbol{k}}, x)dx = P(\xi = (i,j),\ N_m = k_m,\ m = 1, 2, ..., n,\ \ x < \bar{U}_{ij}(t) \le x + dx), \end{aligned}$$

where, as it is stated before, $\boldsymbol{k} = (k_2, ..., k_n)$, $\bar{\boldsymbol{k}} = (k_1, k_2, ..., k_n) \equiv (k_1, \boldsymbol{k})$, and $\bar{U}_{ij}(t)$, $\bar{U}_0(t)$ the elapsed service or vacation time respectively. If finally

$$\begin{aligned} & Q(\boldsymbol{z}) = \textstyle\sum_{\boldsymbol{k} \ge \boldsymbol{0}} q(\boldsymbol{k}) \boldsymbol{z}^{\boldsymbol{k}} \equiv \sum_{k_2 \ge 0} \cdots \sum_{k_n \ge 0} q(k_2, ..., k_n) z_2^{k_2} z_3^{k_3} ... z_n^{k_n}, \\ & P_0(\bar{\boldsymbol{z}}, x) = \textstyle\sum_{\bar{\boldsymbol{k}} \ge \bar{\boldsymbol{0}}} p_0(\bar{\boldsymbol{k}}, x) \bar{\boldsymbol{z}}^{\bar{\boldsymbol{k}}}, \\ & P_{ij}(\bar{\boldsymbol{z}}, x) = \textstyle\sum_{\bar{\boldsymbol{k}} \ge \bar{\boldsymbol{0}}} p_{ij}(\bar{\boldsymbol{k}}, x) \bar{\boldsymbol{z}}^{\bar{\boldsymbol{k}}}, \qquad i, j = 1, 2, \end{aligned}$$

then we arrive easily, for $x > 0$, at

$$\begin{aligned} P_0(\bar{\boldsymbol{z}}, x) = &\ P_0(\bar{\boldsymbol{z}}, 0)(1 - B_0(x)) \exp[-(\lambda - \lambda z_1)x], \\ P_{ij}(\bar{\boldsymbol{z}}, x) = &\ P_{ij}(\bar{\boldsymbol{z}}, 0)(1 - B_{ij}(x)) \exp[-(\lambda - \lambda z_1)x], \end{aligned} \tag{10}$$

and

$$\sum_{j=2}^{n} \mu_j z_j \frac{\partial}{\partial z_j} Q(\boldsymbol{z}) + \lambda Q(\boldsymbol{z}) = P_0((0, \boldsymbol{z}), 0) \beta_0^*(\lambda), \tag{11}$$

with $\beta_0^*(.)$ the LST of $b_0(.)$. For the boundary conditions ($x = 0$) we obtain in a similar way

$$\begin{aligned} P_0((0,\boldsymbol{z}),0) &= \sum_{m=1}^{n-1} p_m z_{m+1} \sum_{i=1}^{m} P_{im}((0,\boldsymbol{z}),0)\beta_{im}^*(\lambda) \\ &\quad + \sum_{i=1}^{n} P_{in}((0,\boldsymbol{z}),0)\beta_{in}^*(\lambda), \end{aligned} \tag{12}$$

$$\begin{aligned} &P_{jj}((0,\boldsymbol{z}),0) = \mu_j \tfrac{d}{dz_j} Q\ (\boldsymbol{z}), && j = 2,...,n, \\ &P_{1j}(\bar{\boldsymbol{z}},0) = P_{11}(\bar{\boldsymbol{z}},0) \textstyle\prod_{m=1}^{j-1} (1-p_m)\beta_{1m}^*(\lambda - \lambda z_1), && j = 2,...,n, \\ &P_{ij}(\bar{\boldsymbol{z}},0) = \ \mu_i \tfrac{d}{dz_i} Q(\boldsymbol{z}) \textstyle\prod_{m=i}^{j-1} (1-p_m)\beta_{im}^*(\lambda - \lambda z_1), && \begin{array}{l} i = 2,...,n \\ j = i+1,...,n, \end{array} \end{aligned} \tag{13}$$

while for the $P_{11}(\bar{\boldsymbol{z}},0)$ we obtain using relations (12) and (13)

$$\begin{aligned} P_{11}(\bar{\boldsymbol{z}},0) &= \{\lambda z_1 Q(\boldsymbol{z}) + \sum_{j=2}^{n} a_j(z_1, z_{j+1},...,z_n) P_{jj}((0,\boldsymbol{z}),0) \\ &\quad - P_0((0,\boldsymbol{z}),0)[1 + \beta_0^*(\lambda) - \beta_0^*(\lambda - \lambda z_1)]\}/[z_1 - a_1(\bar{\boldsymbol{z}})]. \end{aligned} \tag{14}$$

Replacing now in the numerator of (14) the zero $x(\boldsymbol{z})$ of the denominator, we arrive at

$$P_0((0,\boldsymbol{z}),0) = \frac{\lambda x(\boldsymbol{z})Q(\boldsymbol{z}) + \sum_{j=2}^{n} a_j(x(\boldsymbol{z}), z_{j+1},...,z_n) P_{jj}((0,\boldsymbol{z}),0)}{1+\beta_0^*(\lambda) - \beta_0^*(\lambda - \lambda x(\boldsymbol{z}))}, \tag{15}$$

and substituting back from (12)-(15) in (10) and (11) and integrating with respect to x we obtain for $j = 2,...,n$

$$e_{1j}(z_1)P_{1j}(\bar{\boldsymbol{z}}) = \ e_{11}(z_1)P_{11}(\bar{\boldsymbol{z}}) \prod_{m=1}^{j-1} (1-p_m)\beta_{1m}^*(\lambda - \lambda z_1), \tag{16}$$

$$e_{jj}(z_1)P_{jj}(\bar{\boldsymbol{z}}) = \mu_j \frac{d}{dz_j} Q(\boldsymbol{z}), \tag{17}$$

$$e_{ij}(z_1)P_{ij}(\bar{\boldsymbol{z}}) = \ \mu_i \frac{d}{dz_i} Q(\boldsymbol{z}) \prod_{m=i}^{j-1} (1-p_m)\beta_{im}^*(\lambda - \lambda z_1), \quad \begin{array}{l} i = 2,...,n \\ j = i+1,...,n, \end{array} \tag{18}$$

$$e_0(z_1)P_0(\bar{\boldsymbol{z}}) = \ \frac{\lambda x(z_2,...,z_n)Q(\boldsymbol{z}) + \sum_{j=2}^{n} \ e_{jj}(z_1) a_j(x(z_2,...,z_n), z_{j+1},...,z_n) P_{jj}(\bar{\boldsymbol{z}})}{1+\beta_0^*(\lambda) - \beta_0^*(\lambda - \lambda x(\boldsymbol{z}))}, \tag{19}$$

$$\begin{aligned} e_{11}(z_1)P_{11}(\bar{\boldsymbol{z}}) \ &= \{\lambda[z_1 - x(\boldsymbol{z})]Q(\boldsymbol{z}) + \ \textstyle\sum_{j=2}^{n} e_{jj}(z_1)[a_j(z_1, z_{j+1},...,z_n) \\ &\quad - a_j(x(\boldsymbol{z}), z_{j+1},...,z_n)]P_{jj}(\bar{\boldsymbol{z}}) + \ e_0(z_1)[\beta_0^*(\lambda - \lambda z_1) \\ &\quad - \beta_0^*(\lambda - \lambda x(\boldsymbol{z}))]P_0(\bar{\boldsymbol{z}})\}/[z_1 - a_1(\bar{\boldsymbol{z}})], \end{aligned} \tag{20}$$

$$\sum_{j=2}^{n} \mu_j z_j \frac{\partial}{\partial z_j} Q(z) + \lambda Q(z) = e_0(z_1) P_0(\bar{z}) \beta_0^*(\lambda), \tag{21}$$

where in general $P.(\bar{z}) = \int P.(\bar{z}, x)dx$ and

$$e_{ij}(z_1) = \frac{\lambda - \lambda z_1}{1 - \beta_{ij}^*(\lambda - \lambda z_1)}, \qquad e_{ij}(1) = \frac{1}{\bar{b}_{ij}}.$$

We will use in the sequel the expressions above to obtain all generating functions at the point $\bar{z} = \bar{\mathbf{1}}$.

Theorem 3 *For $\rho^* < 1$ the generating functions $P_{ij}(.)$, $P_0(.)$, $Q(.)$ at the point $\bar{z} = \bar{\mathbf{1}}$ are given by*

$$\begin{aligned}
&P_{11}(\bar{\mathbf{1}}) = \lambda \bar{b}_{11}, \qquad P_{jj}(\bar{\mathbf{1}}) = \lambda p_{j-1} \bar{b}_{jj}, \qquad j = 2, ..., n \\
&P_{ij}(\bar{\mathbf{1}}) = \lambda p_{i-1} \bar{b}_{ij} \textstyle\prod_{m=i}^{j-1} (1 - p_m), \qquad i = 1, ..., n,\ j = i+1, ..., n \\
&Q(\mathbf{1}) = \frac{1-\rho^*}{1+\lambda \bar{b}_0 / \beta_0^*(\lambda)}, \\
&P_0(\bar{\mathbf{1}}) = \frac{\lambda \bar{b}_0}{\beta_0^*(\lambda)} [\textstyle\sum_{j=2}^{n} p_{j-1} + Q(\mathbf{1})].
\end{aligned} \tag{22}$$

Proof: Let us define

$$\mathcal{N}(\bar{z}) = Q(z) + P_0(\bar{z}) + \sum_{i=1}^{n} \sum_{j=i}^{n} P_{ij}(\bar{z}), \tag{23}$$

then from relations (16), (17) and (18)

$$P_{ij}(\bar{z}) = \frac{e_{ii}(z_1)}{e_{ij}(z_1)} P_{ii}(\bar{z}) \prod_{m=i}^{j-1} (1 - p_m) \beta_{im}^*(\lambda - \lambda z_1), \qquad i = 1, 2, ..., n,\ j = i+1, ..., n$$

i.e.

$$P_{ij}(\bar{\mathbf{1}}) = P_{ii}(\bar{\mathbf{1}}) \frac{\bar{b}_{ij}}{\bar{b}_{ii}} \prod_{m=i}^{j-1} (1 - p_m), \qquad i = 1, 2, ..., n,\ j = i+1, ..., n \tag{24}$$

and substituting back in (23), using (19) and observing that $\mathcal{N}(\bar{\mathbf{1}}) = \bar{\mathbf{1}}$ we obtain after manipulations

$$\begin{aligned}
&\tfrac{\rho_1}{\lambda \bar{b}_{11}} P_{11}(\bar{\mathbf{1}}) + (1 + \tfrac{\lambda \bar{b}_0}{\beta_0^*(\lambda)}) Q(\mathbf{1}) \\
&\quad + \textstyle\sum_{j=2}^{n} \frac{P_{jj}(\bar{\mathbf{1}})}{\bar{b}_{jj}} [\bar{b}_{jj} + \sum_{k=j+1}^{n} \bar{b}_{jk} \prod_{m=j}^{k-1} (1 - p_m) + \frac{\bar{b}_0}{\beta_0^*(\lambda)}] = 1.
\end{aligned} \tag{25}$$

Using now (19) in (20) and putting $\bar{z} = \bar{\mathbf{1}}$ we arrive at

$$\begin{aligned}
\frac{P_{11}(\bar{\mathbf{1}})}{\bar{b}_{11}} &= \frac{\lambda}{1 - \rho_1} \{ (1 + \frac{\lambda \bar{b}_0}{\beta_0^*(\lambda)}) Q(\mathbf{1}) \\
&\quad + \sum_{j=2}^{n} \frac{P_{jj}(\bar{\mathbf{1}})}{\bar{b}_{jj}} [\bar{b}_{jj} + \sum_{k=j+1}^{n} \bar{b}_{jk} \prod_{m=j}^{k-1} (1 - p_m) + \frac{\bar{b}_0}{\beta_0^*(\lambda)}] \},
\end{aligned} \tag{26}$$

where ρ_1 has been defined in Lemma 1, and so from (24)-(26) above we conclude

$$P_{11}(\bar{1}) = \lambda \bar{b}_{11}, \qquad P_{1j}(\bar{1}) = \lambda \bar{b}_{1j} \prod_{m=1}^{j-1} (1-p_m), \qquad j = 2, 3, ..., n. \tag{27}$$

Multiplying relation (17) by z_j, adding for all j, subtracting from (21) and using (19) we arrive at

$$\frac{\sum_{j=2}^{n} e_{jj}(z_1) P_{jj}(\bar{z})}{Q(z)} [D_j(\bar{z}) - z_j] = \lambda[1 - x(z) f(x(z))], \tag{28}$$

with

$$f(x(z)) = \frac{\beta_0^*(\lambda)}{1 + \beta_0^*(\lambda) - \beta_0^*(\lambda - \lambda x(z))}, \tag{29}$$

$$D_j(z) = f(x(z)) a_j(x(z), z_{j+1}, ..., z_n). \tag{30}$$

Now it is clear from (5), (7) and (29), (30) that

$$D_i(z) \equiv C^{(i)}(0, z),$$

and so for $\rho^* < 1$ Theorem 2 holds for $D_i(z)$.

By putting now $z_1 = 1$ and, for any permutation $(i_2, i_3, ..., i_n)$ of $(2, 3, ..., n)$, replacing z_{i_k} by the corresponding zero $x_{i_k}(z_{i_{k+1}}, ..., z_{i_n})$ we succeed to eliminate all except one terms in the left hand part of (28) and arrive at

$$P_{jj}(\bar{1}) = \frac{\lambda p_{j-1} \bar{b}_{jj}}{1 - \rho^*} (1 + \frac{\lambda \bar{b}_0}{\beta_0^*(\lambda)}) Q(1), \qquad j = 2, 3, ..., n, \tag{31}$$

and replacing in (25) we obtain after manipulations

$$Q(1) = \frac{1 - \rho^*}{1 + \frac{\lambda \bar{b}_0}{\beta_0^*(\lambda)}}, \tag{32}$$

which is the third of (22).

From (32) and (31) we obtain the first of (22) and putting back $P_{jj}(\bar{1})$ in (24) and (19) we arrive at the second and forth of (22) respectively and the theorem has been proved. □

5 Mean number of ordinary and retrial customers

For $p_0 \equiv 1$ and $j = 1, ..., n$, let us define now

$$\begin{aligned} \pi_j = \; & p_{j-1} \{ \bar{b}_{jj}^{(2)} + \sum_{k=j+1}^{n} \bar{b}_{jk}^{(2)} \prod_{m=j}^{k-1} (1-p_m) \\ & + 2 \sum_{r=j}^{n-1} \prod_{m=j}^{r} (1-p_m) \bar{b}_{jr} [\bar{b}_{jr+1} + \sum_{k=r+2}^{n} \bar{b}_{jk} \prod_{m=r+1}^{k-1} (1-p_m)] \}, \end{aligned} \tag{33}$$

then, for the mean length of the ordinary queue,

Theorem 4 *The mean number of P_1 customers, in the ordinary queue is given by,*

$$E(N_1) = \frac{\lambda^2}{2(1-\rho_1)}\{\frac{\bar{b}_0}{\beta_0^*(\lambda)}[\sum_{k=2}^{n} p_{k-1} + Q(\mathbf{1})] + \sum_{k=1}^{n} \pi_k\}, \qquad (34)$$

where $Q(\mathbf{1})$ and π_k are given by (32), (33) respectively.

Proof: Differentiating relations (17), (18), (19), with respect to z_1, and setting $z_l = 1,\ l = 1, 2, ..., n$, we arrive easily at,

$$\begin{aligned}
\frac{\partial P_{ii}(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}} &= \frac{\lambda^2 p_{i-1}\bar{b}_{ii}^{(2)}}{2}, \\
\frac{\partial P_{ij}(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}} &= \frac{\lambda^2 p_{i-1}\bar{b}_{ij}^{(2)}}{2}\prod_{m=i}^{j-1}(1-p_m) + \lambda^2 p_{i-1}\bar{b}_{ij}\prod_{m=i}^{j-1}(1-p_m)\sum_{m=i}^{j-1}\bar{b}_{im}, \qquad (35)\\
& \qquad i = 2, 3, ..., n,\ j = i+1, ..., n, \\
\frac{\partial P_0(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}} &= \frac{\lambda^2 \bar{b}_0^{(2)}}{2\beta_0^*(\lambda)}[\sum_{k=2}^{n} p_{k-1} + Q(\mathbf{1})].
\end{aligned}$$

In a similar way, from (20), (16),

$$\begin{aligned}
\frac{\partial P_{11}(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}} &= \frac{\lambda^3 \bar{b}_{11}}{2(1-\rho_1)}[\frac{\bar{b}_0^{(2)}}{\beta_0^*(\lambda)}[\sum_{k=2}^{n} p_{k-1} + Q(\mathbf{1})] + \sum_{k=1}^{n}\pi_k] + \frac{\lambda^2\bar{b}_{11}^{(2)}}{2}, \\
\frac{\partial P_{1j}(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}} &= \frac{\lambda^3 \bar{b}_{1j}}{2(1-\rho_1)}\prod_{m=1}^{j-1}(1-p_m)[\frac{\bar{b}_0^{(2)}}{\beta_0^*(\lambda)}[\sum_{k=2}^{n} p_{k-1} + Q(\mathbf{1})] + \sum_{k=1}^{n}\pi_k] \\
&+ \lambda^2\prod_{m=1}^{j-1}(1-p_m)[\bar{b}_{1j}\sum_{m=1}^{j-1}\bar{b}_{1m} + \frac{\bar{b}_{1j}^{(2)}}{2}], \qquad j = 2, 3, ..., n.
\end{aligned} \qquad (36)$$

Observing now that

$$E(N_1) = \frac{\partial \mathcal{N}(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}} = \sum_{i=1}^{n}\sum_{j=i}^{n}\frac{\partial P_{ij}(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}} + \frac{\partial P_0(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}},$$

and replacing from (35) and (36) we obtain relation (34) and the theorem has been proved. □

Before giving the mean queue lengths for the retrial customers we have to state some preliminary results. Let, for $k, j = 2, ..., n,\ m = 1, 2,$

$$h_k^{(m)} = \frac{d^m x(z)}{dz_k^m}|_{z=1}, \qquad \hat{\rho}_{jk}^{(m)} = \frac{d^m \alpha_j(x(z), ..., z_k, ...1)}{dz_k^m}|_{z=1},$$

where $x(z)$ is defined in Lemma 1 (note that $\hat{\rho}_{jk}^{(u)} \neq \hat{\rho}_{kj}^{(u)},\ k \neq j$). Then after some algebra we obtain,

$$\begin{aligned}
h_k^{(1)} &= \tfrac{p_{k-1}}{(1-\rho_1)} \prod_{m=1}^{k-2}(1-p_m),\\
h_k^{(2)} &= \tfrac{\lambda^2 (h_k^{(1)})^2}{(1-\rho_1)}\pi_1 + 2\lambda (h_k^{(1)})^2 \sum_{m=1}^{k-1} \bar{b}_{1m},\\
\hat{\rho}_{jk}^{(1)} &= \lambda h_k^{(1)}(\bar{b}_{jj} + \textstyle\sum_{r=j+1}^{n} \bar{b}_{jr} \prod_{m=j}^{r-1}(1-p_m)) + \delta_{\{k>j\}}[p_{k-1}\prod_{m=j}^{k-2}(1-p_m)],\\
\hat{\rho}_{jk}^{(2)} &= \lambda h_k^{(2)}(\bar{b}_{jj} + \textstyle\sum_{r=j+1}^{n} \bar{b}_{jr} \prod_{m=j}^{r-1}(1-p_m)) + \tfrac{(\lambda h_k^{(1)})^2}{p_{j-1}}\pi_j\\
&\quad +\delta_{\{k>j\}} 2\lambda h_k^{(1)} p_{k-1} \textstyle\prod_{m=j}^{k-2}(1-p_m) \sum_{r=j}^{k-1} \bar{b}_{jr}, \qquad k,j = 2,...,n.
\end{aligned}$$

with $\prod_{m=j}^{i}(1-p_m) = 1$ for $j > i$. Moreover define

$$\begin{aligned}
s_k = &\ \tfrac{\lambda Q(1,...,1)h_k^{(2)}}{2p_{k-1}\prod_{m=1}^{k-2}(1-p_m)} + \tfrac{\lambda^2 p_{k-1}}{\mu_k(1-\rho_1)} + \tfrac{P_0(1,...,1)}{2p_{k-1}\bar{b}_0 \prod_{m=1}^{k-2}(1-p_m)}[\lambda \bar{b}_0 h_k^{(2)} + (\lambda h_k^{(1)})^2 \bar{b}_0^{(2)}]\\
&+ \tfrac{\lambda}{2p_{k-1}\prod_{m=1}^{k-2}(1-p_m)} \sum_{j=2}^{n} p_{j-1}\hat{\rho}_{jk}^{(2)} + \tfrac{\lambda^2 \bar{b}_0}{(1-\rho_1)\beta_0^*(\lambda)}[(1-\rho^*)h_k^{(1)} + \tfrac{\lambda p_{k-1}}{\mu_k}]\\
&+ \tfrac{\lambda}{(1-\rho_1)} \sum_{j=2}^{n} \rho_{0j}(\hat{\rho}_{jk}^{(1)} + \tfrac{\lambda h_k^{(1)}\bar{b}_0}{\beta_0^*(\lambda)}), \qquad k = 2,...,n.
\end{aligned}$$

and denote

$$D_{kl} = \frac{\partial^2}{\partial z_k \partial z_l} Q(z)|_{z=1}, \qquad k,l = 2,3,...,n.$$

Note here that $D_{kl} = D_{lk}\ \forall\ k,l$. Then we state the following theorem.

Theorem 5 *The quantities $D_{kl},\ k,\ l = 2,...,n$ can be found as the solution of the system of linear equations,*

$$\begin{aligned}
(\mu_k + \mu_l)D_{kl} = &\ \sum_{j=2}^{n-1} \mu_j D_{kj}(p_{l-1}\prod_{m=j}^{l-2}(1-p_m))\delta_{\{l>j\}} + \sum_{j=2}^{n-1} \mu_j D_{lj}\delta_{\{k>j\}}\\
&\times(p_{k-1}\prod_{m=j}^{k-2}(1-p_m)) + p_{l-1}\prod_{m=1}^{l-2}(1-p_m)[\sum_{j=2}^{n} \tfrac{\mu_j(\rho_j+\rho_{0j})}{p_{j-1}(1-\rho_1)} D_{kj} + s_k]\\
&+p_{k-1}\prod_{m=1}^{k-2}(1-p_m)[\sum_{j=2}^{n} \tfrac{\mu_j(\rho_j+\rho_{0j})}{p_{j-1}(1-\rho_1)} D_{lj} + s_l].
\end{aligned} \tag{37}$$

Proof: Replacing (19) to (20) we arrive at

$$e_0(z_1)P_0(\bar{z}) = \frac{\lambda z_1 Q(z) + \sum_{j=2}^{n} e_{jj}(z_1)a_j(z_1,z_{j+1},...,z_n)P_{jj}(\bar{z}) + e_{11}(z_1)P_{11}(\bar{z})(a_1(\bar{z})-z_1)}{1+\beta_0^*(\lambda)-\beta_0^*(\lambda-\lambda z_1)},$$

and replacing to (21) and setting $z_1 = 1$ we obtain

$$\sum_{j=2}^{n} \mu_j z_j \frac{\partial}{\partial z_j} Q(z) = \sum_{j=2}^{n-1} \frac{P_{jj}(1,z)}{\bar{b}_{jj}} a_j(1, z_{j+1}, ..., z_n) + \frac{P_{11}(1,z)}{\bar{b}_{11}}(a_1(1,z) - 1). \quad (38)$$

Now adding relation (17) for all $j = 2, ..., n$, putting $z_1 = 1$ and subtracting from (38) we arrive at our basic equation,

$$\sum_{j=2}^{n} \mu_j (z_j - 1) \frac{\partial}{\partial z_j} Q(z) = \sum_{j=1}^{n-1} \frac{P_{jj}(1,z)}{\bar{b}_{jj}} [a_j(1, z_{j+1}, ..., z_n) - 1]. \quad (39)$$

Taking finally derivatives above with respect to z_k, z_l, and using the fact that from (17) $\frac{\partial P_{jj}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} = \mu_j D_{jk} \bar{b}_{jj}$, $\forall\ k,\ j = 2, 3, ..., n$, we arrive after some algebra at relation (37). □

Now we are ready to give the mean number of customers in the retrial boxes.

Lemma 6 *The mean number of P_k, $k = 2, 3, ..., n$ customers in K_{k-1}^{th} retrial box is given by*

$$\begin{aligned} E(N_k) = & \sum_{m=2}^{n} \frac{\mu_m(\rho_m + \rho_{0m})}{\lambda p_{m-1}(1-\rho_1)} D_{mk} + \frac{s_k \rho_1}{\lambda} + \sum_{m=2}^{n} \rho_{0m} (\hat{\rho}_{mk}^{(1)} + \frac{\lambda h_k^{(1)} \bar{b}_0}{\beta_0^*(\lambda)}) \\ & + \frac{\lambda \bar{b}_0}{\beta_0^*(\lambda)} [(1-\rho^*) h_k^{(1)} + \frac{\lambda p_{k-1}}{\mu k}] + \frac{\lambda p_{k-1}}{\mu_k}. \end{aligned} \quad (40)$$

Proof: Differentiating (20), (16), with respect to z_k, and setting $z_l = 1$, $l = 1, 2, ..., n$, we obtain after manipulations,

$$\begin{aligned} \frac{\partial P_{11}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} = & \bar{b}_{11} [\sum_{m=2}^{n} \frac{\mu_m(\rho_m + \rho_{0m})}{p_{m-1}(1-\rho_1)} D_{mk} + s_k], \\ \frac{\partial P_{1j}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} = & \frac{\bar{b}_{1j} \prod_{r=1}^{j-1}(1-p_r)}{\bar{b}_{11}} \frac{\partial P_{11}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}}, \quad j = 2, ..., n. \end{aligned} \quad (41)$$

In a similar way from (17), (18), (19),

$$\begin{aligned} \frac{\partial P_{ii}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} = & \mu_i D_{ik} \bar{b}_{ii}, \\ \frac{\partial P_{ij}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} = & \frac{\bar{b}_{ij} \prod_{r=i}^{j-1}(1-p_r)}{\bar{b}_{ii}} \frac{\partial P_{ii}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}}, \quad \begin{array}{l} i = 2, ..., n, \\ j = i+1, ..., n, \end{array} \\ \frac{\partial P_0(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} = & \sum_{m=2}^{n} \frac{\mu_m \rho_{0m}}{\lambda p_{m-1}} D_{mk} + \frac{\lambda \bar{b}_0}{\beta_0^*(\lambda)} [(1-\rho^*) h_k^{(1)} + \frac{\lambda p_{k-1}}{\mu_k}] \\ & + \sum_{m=2}^{n} \rho_{0m} (\hat{\rho}_{mk}^{(1)} + \frac{\lambda h_k^{(1)} \bar{b}_0}{\beta_0^*(\lambda)}). \end{aligned} \quad (42)$$

Observing now that, for any $k = 2, 3, ..., n$,

$$E(N_k) = \frac{\partial \mathcal{N}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} = \frac{\partial Q(z)}{\partial z_k}|_{z=1} + \sum_{i=1}^{n} \sum_{j=i}^{n} \frac{\partial P_{ij}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} + \frac{\partial P_0(\bar{z})}{z_k}|_{\bar{z}=\bar{1}}, \quad (43)$$

and replacing from (41), (42) to (43) we arrive easily at (40) and the theorem has been proved. □

6 Numerical Results

In this section we consider a system of $n = 4$ phases of service and so three retrial boxes and use the formulae derived previously to obtain numerical results and to investigate the way the mean number of customers in the retrial boxes $E(N_i)$ $i = 2, 3, 4$ are affected when we vary the mean vacation time $\bar{b}_0$, the mean service time of the ordinary customers $\bar{b}_{11}$, and the mean retrial interval in the first box $E(retrial\ K_1) = 1/\mu_1$, always for increasing values of the mean arrival rate λ.

To construct the tables we assume that the vacation time U_0 and the service times follow exponential distributions with p.d.f.'s respectively,

$$b_0(x) = \frac{1}{\bar{b}_0} e^{-(1/\bar{b}_0)x}, \quad b_{ij}(x) = \frac{1}{\bar{b}_{ij}} e^{-(1/\bar{b}_{ij})x}, \quad \begin{array}{l} i = 1, ..., 4 \\ j = i, ..., 4. \end{array}$$

Moreover we assume that in all tables below $\bar{b}_{14} = 0.2$, $\bar{b}_{12} = \bar{b}_{22} = \bar{b}_{23} = \bar{b}_{33} = 0.33$, $\bar{b}_{13} = \bar{b}_{24} = \bar{b}_{34} = \bar{b}_{44} = 0.25, p_1 = 0.7$, $p_2 = 0.5$, $p_3 = 0.1$. Finally $\mu_2 = 0.5$, $\mu_3 = 2$.

Table 1 shows the way $E(N_i)$ $i = 2, 3, 4$ changes when we vary the mean vacation time, for increasing values of the mean arrival rate λ. Here one can observe the crucial role that the vacation plays on the number of retrial customers. Thus, even for a small value of λ, $\lambda = 0.15$ for example, $E(N_2)$ increases from 0.1488 to 46.026 when we pass from a system without vacation period ($\bar{b}_0 = 0$) to the system with $\bar{b}_0 = 2.7$, while the corresponding value for $E(N_3)$ increases from 0.2029 to 64.83. When now the arrival rate λ becomes $\lambda = 0.42$ then even a small change from $\bar{b}_0 = 0$ to $\bar{b}_0 = 0.6$ increases dramatically the mean number of retrial customers to 267.02 in retrial box K_1 and to 380.45 in retrial box K_2 respectively. Thus we must be very careful on the vacation period that we must allow, to avoid overcrowded retrial boxes. The behavior of the third retrial box K_3 ($E(N_4)$) is smoother, and it shows us a way to reduce this dramatic effect of the vacation period by allowing faster retrials ($E(retrial\ K_3) = 1/\mu_3 = 0.5$

here) and/or less preference of the box ($p_3 = 0.1$).

$\lambda \backslash \bar{b}_0$		0	0.2	0.6	1.3	2.7
0.15	$E(N_2)$	0.1488	0.1926	0.3084	0.6816	46.026
	$E(N_3)$	0.2029	0.2451	0.3602	0.7605	64.83
	$E(N_4)$	0.0112	0.0161	0.0282	0.0623	3.316
0.27		0.3798	0.5273	1.1548	120.04	
		0.4958	0.6693	1.3906	170.57	
		0.0287	0.0435	0.0954	8.599	
0.42		1.045	1.8373	267.02		
		1.3022	2.2913	380.45		
		0.0768	0.1395	19.077		
0.59		4.1926	221.4			
		5.2556	314.4			
		0.2956	15.814			
0.71		126.8				
		179.3				
		9.0327				

Table 1 : *Values of* $E(N_i)$, $i = 2, 3, 4$, *for* $\mu_1 = 1$, $\bar{b}_{11} = 0.5$.

Similar observations can be deduced from Table 2 that contains values of $E(N_i)$ $i = 2, 3, 4$ when we vary the mean first stage service $\bar{b}_{11}$. One can observe again the way the mean number of retrial customers in each box increases when $\bar{b}_{11}$ increases. An increase that depends on how fast or slow the mean retrial $E(retrial\ K_i)$ is and/or on the preference that customers show to the corresponding box p_i.

$\lambda \backslash \bar{b}_{11}$		0.2	0.8	1.3	2.1	2.8	5.5
0.15	$E(N_2)$	0.1711	0.2233	0.3014	0.5481	1.0297	18522.3
	$E(N_3)$	0.2257	0.271	0.3291	0.4882	0.7737	26410.6
	$E(N_4)$	0.0145	0.0181	0.0226	0.0346	0.0595	1321.15
0.25		0.3646	0.6088	1.1043	4.3585	193.69	
		0.4783	0.7108	1.1131	3.97	263.19	
		0.0306	0.047	0.0755	0.25	13.378	
0.3		0.5051	0.9877	2.267	64.9		
		0.662	1.1428	2.354	85.09		
		0.042	0.0742	0.1496	4.423		
0.4		0.9447	3.034	112.03			
		1.2382	3.6085	156.15			
		0.0765	0.2161	7.9094			
0.5		1.8448	108.39				
		2.4349	152.81				
		0.1444	7.717				
0.71		92.598					
		131.62					
		6.632					

Table 2 : Values of $E(N_i)$, $i = 2, 3, 4$, *for* $\mu_1 = 1$, $\bar{b}_0 = 0.2$.

Table 3 finally depicts the way the $E(N_i)$ $i = 2, 3, 4$ are affected when we vary the mean retrial rate in the first box $E(retrial) = 1/\mu_1$. One can observe here not only the increase of $E(N_2)$, but mainly the reduction of $E(N_3)$ and $E(N_4)$ in the other two boxes when we increase $1/\mu_1$, a reduction which is more apparent when λ increases.

$\lambda \backslash E(retrial\ K_1)$		0.02	0.2	1	2	10
0.15	$E(N_2)$	0.0452	0.0727	0.1926	0.3397	1.4994
	$E(N_3)$	0.2539	0.2515	0.2454	0.2409	0.2329
	$E(N_4)$	0.0177	0.0171	0.0161	0.0157	0.0151
0.27		0.1423	0.2185	0.5373	0.9183	3.862
		0.7229	0.7087	0.6693	0.6443	0.5987
		0.0521	0.0488	0.0435	0.413	0.0382
0.42		0.4837	0.761	1.8373	3.045	11.953
		2.7032	2.5895	2.2913	2.1153	1.8179
		0.1904	0.1699	0.1395	0.1271	0.1093
0.59		8.33	60.25	221.4	371.7	1347.5
		464.3	419.03	314.4	265.23	192.75
		23.5	21.12	15.814	13.31	9.6763

Table 3 : Values of $E(N_i)$, $i = 2, 3, 4$ *for* $\bar{b}_0 = 0.2$, $\bar{b}_{11} = 0.5$.

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On p-Adic Vector Measure Spaces

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Abstract

For $\mathcal{R}$ be a separating algebra of subsets of a set X, E a complete Hausdorff non-Archimedean locally convex space and $m : \mathcal{R} \to E$ a bounded finitely additive measure, we study some of the properties of the integrals with respect to m of scalar valued functions on X. The concepts of convergence in measure, with respect to m, and of m-measurable functions are introduced and several results concerning these notions are given.

1 Preliminaries

Throughout this paper, $\mathbb{K}$ will be a complete non-Archimedean valued field, whose valuation is non-trivial. By a seminorm, on a vector space over $\mathbb{K}$, we will mean a non-Archimedean seminorm. Similarly, by a locally convex space we will mean a non-Archimedean locally convex space over $\mathbb{K}$ (see [10] or [11]). For E a locally convex space, we will denote by cs(E) the collection of all continuous seminorms on E. For X a set, $f \in \mathbb{K}^X$ and $A \subset X$, we define

$$\|f\|_A = \sup\{|f(x)| : x \in A\} \quad \text{and} \quad \|f\| = \|f\|_X.$$

Also for $A \subset X$, A^c will be its complement in X and χ_A the $\mathbb{K}$-valued characteristic function of A. The family of all subsets of X will be denoted by $P(X)$.

Assume next that X is a non-empty set and $\mathcal{R}$ a separating algebra of subsets of X, i.e. $\mathcal{R}$ is a family of subsets of X such that

1. $X \in \mathcal{R}$, and, if $A, B \in \mathcal{R}$, then $A \cup B$, $A \cap B$, A^c are also in $\mathcal{R}$.

2. If x, y are distinct elements of X, then there exists a member of $\mathcal{R}$ which contains x but not y.

Then $\mathcal{R}$ is a base for a Hausdorff zero-dimensional topology $\tau_{\mathcal{R}}$ on X. For E a locally convex space, we denote by $M(\mathcal{R}, E)$ the space of all finitely-additive measures $m : \mathcal{R} \to E$ such that $m(\mathcal{R})$ is a bounded subset of E (see [7]). For a net (V_δ) of subsets of X, we write $V_\delta \downarrow \emptyset$ if (V_δ) is decreasing and $\cap V_\delta = \emptyset$. An element $m \in M(\mathcal{R}, E)$ is said to be σ-additive if $m(V_n) \to 0$ for each sequence (V_n) in $\mathcal{R}$ which decreases to the empty set. We denote by $M_\sigma(\mathcal{R}, E)$ the space of all σ-additive members of $M(\mathcal{R}, E)$. An m of $M(\mathcal{R}, E)$ is said to be τ-additive if $m(V_\delta) \to 0$ for each net (V_δ) in $\mathcal{R}$ with $V_\delta \downarrow \emptyset$. We will denote by $M_\tau(\mathcal{R}, E)$ the space of all τ-additive members of $M(\mathcal{R}, E)$. For $m \in M(\mathcal{R}, E)$ and $p \in cs(E)$, we define

$$m_p : \mathcal{R} \to \mathbb{R}, \quad m_p(A) = \sup\{p(m(V)) : V \in \mathcal{R}, V \subset A\} \quad \text{and} \quad \|m\|_p = m_p(X).$$

We also define

$$N_{m,p} : X \to \mathbb{R}, \quad N_{m,p}(x) = \inf\{m_p(V) : x \in V \in \mathcal{R}\}.$$

Next we will recall the definition of the integral of an $f \in \mathbb{K}^X$ with respect to some $m \in M(\mathcal{R}, E)$. Assume that E is a complete Hausdorff locally convex space. For $A \subset X$, let $\mathcal{D}_A$ be the family of all $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\}$, where $\{A_1, A_2, \ldots, A_n\}$ is an $\mathcal{R}$-partition of A and $x_k \in A_k$. We make $\mathcal{D}_A$ into a directed set by defining $\alpha_1 \geq \alpha_2$ if the partition of A in α_1 is a refinement of the one in α_2. For $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\}$, we define $\omega_\alpha(f, m) = \sum_{k=1}^n f(x_k) m(A_k)$. If the limit $\lim \omega_\alpha(f, m)$ exists in E, we will say that f is m-integrable over A and denote this limit by $\int_A f\, dm$. For $A = X$, we write simply $\int f\, dm$. It is easy to see that if f is m-integrable over X, then it is m-integrable over every $A \in \mathcal{R}$ and $\int_A f\, dm = \int \chi_A f\, dm$. If f is bounded on A, then

$$p\left(\int_A f\, dm\right) \leq \|f\|_A \cdot m_p(A).$$

2 Measurable Sets

Throughout the paper, $\mathcal{R}$ will be a separating algebra of subsets of a set X, E a complete Hausdorff locally convex space and $M(\mathcal{R}, E)$ the space of all bounded E-valued finitely-additive measures on $\mathcal{R}$. We will denote by $\tau_{\mathcal{R}}$ the topology on X which has $\mathcal{R}$ as a basis. Every member of $\mathcal{R}$ is $\tau_{\mathcal{R}}$-clopen, i.e both closed and open. By $S(\mathcal{R})$ we will denote the space of all $\mathbb{K}$-valued $\mathcal{R}$-simple functions. As in [7], if $m \in M(\mathcal{R}, E)$, then a subset A of X is said to be m-measurable if the characteristic function χ_A is m-integrable. By [7, Theorem 4.7], A is measurable iff, for each $p \in cs(E)$ and each $\epsilon > 0$, there

exist V, W in $\mathcal{R}$ such that $V \subset A \subset W$ and $m_p(W \setminus V) < \epsilon$.
Let $\mathcal{R}_m$ be the family of all m-measurable sets. By [7] we have the following

Theorem 2.1 *1. $\mathcal{R}_m$ is an algebra of subsets of X.*

2. *If $\bar{m} : \mathcal{R}_m \to E$, $\bar{m}(A) = \int \chi_A \, dm$, then $\bar{m} \in M(\mathcal{R}_m, E)$.*

3. *$\bar{m}$ is σ-additive iff m is σ-additive.*

4. *$\bar{m}$ is τ-additive iff m is τ-additive.*

5. *For $p \in cs(E)$, we have $N_{m,p} = N_{\bar{m},p}$.*

6. *$\mathcal{R}_m = \mathcal{R}_{\bar{m}}$.*

7. *For $A \in \mathcal{R}$, we have $m_p(A) = \bar{m}_p(A)$.*

8. *For $A \in \mathcal{R}_m$, we have*

$$\bar{m}_p(A) = \inf\{m_p(W) : W \in \mathcal{R}, A \subset W\}.$$

9. *If $f \in \mathbb{K}^X$ is m-integrable, then f is $\bar{m}$-integrable and $\int f \, dm = \int f \, d\bar{m}$.*

10. *If f is bounded and $\bar{m}$-integrable, then f is m-integrable.*

11. *An $f \in \mathbb{K}^X$ is m-integrable iff, for each $p \in cs(E)$ and each $\epsilon > 0$, there exists an $\mathcal{R}$-partition $\{A_1, \ldots, A_n\}$ of X such that, for each $1 \le k \le n$, we have $|f(x) - f(y)| \cdot m_p(A_k) \le \epsilon$ if $x, y \in A_k$. In this case, if $x_k \in A_k$, then*

$$p\left(\int f \, dm - \sum_{k=1}^{n} f(x_k) m(A_k)\right) \le \epsilon.$$

12. *If m is τ-additive, then a subset A of X is measurable iff A is $\tau_{\mathcal{R}_m}$-clopen.*

For $m \in M(\mathcal{R}, E)$ and $p \in cs(E)$, we define

$$m_p^* : P(X) \to \mathbb{R}, \quad m_p^*(A) = \inf\{m_p(W) : A \subset W \in \mathcal{R}\}.$$

It is easy to see that

$$m_p^*(A \cup B) = \max\{m_p^*(A), m_p^*(B)\}.$$

By [7, Theorem 4.10], we have that $m_p^*(A) = \bar{m}_p(A)$ for all $A \in \mathcal{R}_m$.
For $p \in cs(E)$, define

$$d_p : P(X) \times P(X) \to \mathbb{R}, \quad d_p(A, B) = m_p^*(A \triangle B),$$

where $A\triangle B = (A\backslash B)\bigcup(B\backslash A)$. It is easy to see that d_p is an ultrapseudometric on $P(X)$. Let $\mathcal{U}_m$ be the uniformity induced by the family of pseudometrics d_p, $p \in cs(E)$.
For A, B in $\mathcal{R}$, we have

$$p(m(A) - m(B)) \leq m_p(A\triangle B) = m_p(A, B).$$

Hence $m : \mathcal{R} \to E$ is $\mathcal{U}_m$-uniformly continuous. Let G_m be the closure of $\mathcal{R}$ in $(P(X), \mathcal{U}_m)$. Then m has a unique uniformly continuous extension $\hat{m} : G_m \to E$.

Theorem 2.2 *$G_m = \mathcal{R}_m$ and $\hat{m} = \bar{m}$.*

Proof : Assume that $A \in G_m$ and let $p \in cs(E)$ $\epsilon > 0$. There exists $V_1 \in \mathcal{R}$ such that $m_p^*(A\triangle V_1) < \epsilon$. Let W_1 in $\mathcal{R}$ be such that $A\triangle V_1 \subset W_1$ and $m_p(W_1) < \epsilon$. Let $V = V_1 \cap W_1^c$, $W = V_1 \cup W_1$. Then $V \subset A \subset W$. Moreover, $W\backslash V = W_1$, and so $m_p(W \setminus V) < \epsilon$, which proves that $A \in \mathcal{R}_m$. Conversely, suppose that $A \in \mathcal{R}_m$ and let V, W in $\mathcal{R}$ be such that $V \subset A \subset W$ and $m_p(W \setminus V) < \epsilon$. Since $A\triangle V = A \setminus V \subset W \setminus V$, we have that $m_p^*(A\triangle V) \leq m_p(W \setminus V) < \epsilon$, which proves that $A \in G_m$. Finally, for A, B in $\mathcal{R}_m$, we have

$$p(\bar{m}(A) - \bar{m}(B)) = p(\bar{m}(A\triangle B)) \leq \bar{m}_p(A\triangle B) = d_p(A, B).$$

Hence $\bar{m}$ is a $\mathcal{U}_m$-uniformly continuous extension of m and so $\bar{m} = \hat{m}$. This completes the proof.

Definition 2.3 *If $m \in M(\mathcal{R}, E)$, then a subset A of X is said to be m-negligible if $m_p^*(A) = 0$ for every $p \in cs(E)$. A property concerning elements of X is said to be true almost everywhere with respect to m (in short m-a.e) if the set of all points in X for which it is false is m-negligible.*

It is clear that every m-negligible set is measurable.

Theorem 2.4 *Let $m \in M_\sigma(\mathcal{R}, E)$ and suppose that $\mathcal{R}$ is a σ-algebra. Then :*

1. *A subset B of X is measurable iff, for each $p \in cs(E)$, there are $V, W \in \mathcal{R}$ with $V \subset B \subset W$ and $m_p(V) = m_p(W) = m_p^*(B)$, $m_p(W \setminus V) = 0$.*

2. *$\mathcal{R}_m$ is a σ-algebra.*

3. *If E is metrizable, then B is measurable iff there are a $V \in \mathcal{R}$ and an m-negligible set A such that $B = A \cup V$.*

Proof : 1. Suppose that B s measurable. There are an increasing sequence (V_n) in $\mathcal{R}$ and a decreasing sequence (W_n) in $\mathcal{R}$ such that $V_n \subset B \subset W_n$

and $m_p(W_n \setminus V_n) < 1/n$. Let $V = \bigcup V_n$, $W = \bigcap W_n$. Then $V, W \in \mathcal{R}$ and $m_p(W \setminus V) = 0$. Since $B = V \bigcup (B \setminus V) \subset V \bigcup (W \setminus V)$, we have that

$$m_p^*(B) = \bar{m}_p(B) \leq \max\{m_p(V), m_p(W \setminus V)\} = m_p(V) \leq m_p^*(B)$$

and so $m_p^*(B) = m_p(V)$. Analogously we prove that $m_p(W) = m_p^*(B)$.
2. Let (A_n) be a sequence in $\mathcal{R}_m$, $A = \bigcup A_n$, $p \in cs(E)$ and $\epsilon > 0$. For each n, there are $V_n, W_n \in \mathcal{R}$ with $V_n \subset A_n \subset W_n$ and $m_p(W_n \setminus V_n) < \epsilon$. The sets $V = \bigcup V_n$, $W = \bigcup W_n$ are in $\mathcal{R}$ and $W \setminus V \subset \bigcup_{n=1}^{\infty} W_n \setminus V_n$, and therefore $m_p(W \setminus V) \leq \sup_n m_p(W_n \setminus V_n) \leq \epsilon$. This proves that $A \in \mathcal{R}_m$.
3. Suppose that E is metrizable and let (p_n) be an increasing sequence of continuous seminorms on E such that, for each $p \in cs(E)$, there exists n with $p \leq p_n$. Assume that B is measurable. For each n, there are $V_n, W_n \in \mathcal{R}$ with $V_n \subset B \subset W_n$ and $m_{p_n}(W_n \setminus V_n) = 0$. Let $V = \bigcup V_n$, $W = \bigcap W_n$. Then $V, W \in \mathcal{R}$. Given $p \in cs(E)$, there exists n such that $p \leq p_n$ and so

$$m_p(W \setminus V) \leq m_{p_n}(W \setminus V) \leq m_{p_n}(W_n \setminus V_n) = 0.$$

The set $A = B \setminus V \subset W \setminus V$ is m-negligible and $B = V \cup A$. Hence the result follows.

Theorem 2.5 *Let $m \in M_\sigma \mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let (A_n) be a sequence of measurable subsets of Xwhich converges to some A in $P(X)$ with respect to the topology induced by the uniformity $\mathcal{U}_m$. Let*

$$B_1 = \liminf A_n = \bigcup_n \bigcap_{k \geq n} A_k, \quad B_2 = \limsup A_n = \bigcap_n \bigcup_{k \geq n} A_k.$$

Then A is measurable and the sets $B_2 \setminus B_1$, $A \triangle B_1$ and $A \triangle B_2$ are m-negligible. Moreover $A_n \to B_1$ and $A_n \to B_2$.

Proof : Since $\mathcal{R}_m$ is closed in $P(X)$, it follows that A is measurable. Let $p \in cs(E)$ and $\epsilon > 0$. There exists n_o such that $\bar{m}_p(A \triangle A_n) < \epsilon$ for all $n \geq n_o$. Since

$$A \setminus B_2 \subset A \setminus B_1 = \bigcap_n \bigcup_{k \geq n} A \setminus A_k,$$

we have that

$$\bar{m}_p(A \setminus B_2) \leq \bar{m}_p(A \setminus B_1) \leq \bar{m}_p\left(\bigcup_{k \geq n_o} (A \setminus A_k)\right) = \sup_{k \geq n_o} \bar{m}_p(A \setminus A_k) \leq \epsilon.$$

Also

$$B_1 \setminus A \subset B_2 \setminus A = \bigcap_n \left(\bigcup_{k \geq n} A_k \setminus A\right) \subset \bigcup_{k \geq n_o} (A_k \setminus A)$$

and so $\bar{m}_p(B_1 \setminus A) \leq \bar{m}_p(B_2 \setminus A) \leq \epsilon$. This, being true for each $\epsilon > 0$, implies that the sets $B_1 \triangle A$ and $B_2 \triangle A$ are m-negligible. Moreover $B_1 \triangle B_2 \subset (B_1 \triangle A) \bigcup (B_2 \triangle A)$, and so $B_1 \triangle B_2$ is m-negligible. Finally,

$$A_n \triangle B_1 \subset (A_n \triangle A) \cup (A \triangle B_1)$$

and so $\bar{m}_p(A_n \triangle B_1 \leq \bar{m}_p(A_n \triangle A) \to 0$, which proves that $A_n \to B_1$. Similarly $A_n \to B_2$.

Theorem 2.6 *Let $m \in M_\sigma(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let $f \in \mathbb{K}^X$. Then, f ia m-integrable iff it is $\bar{m}$-integrable. Moreover*

$$\int f\, dm = \int f\, d\bar{m}.$$

Proof : By Theorem 2.1, if f is m-integrable, then it is also $\bar{m}$-integrable and the two integrals coincide. Conversely, suppose that f is $\bar{m}$-integrable and let $p \in cs(E)$ and $\epsilon > 0$. By Theorem 2.1, there exists an $\mathcal{R}_m$-partition $\{A_1, \ldots, A_n\}$ of X such that, for each $k = 1, 2, \ldots$, we have $|f(x) - f(y)| \cdot \bar{m}_p(A_k) < \epsilon$ if $x, y \in A_k$. In view of Theorem 2.4, there are sets $V_k, W_k \in \mathcal{R}$ with $V_k \subset A_k \subset W_k$ and $m_p(W_k \setminus V_k) = 0$, $m_p(V_k) = \bar{m}_p(A_k)$. Let $V_{n+1} = X \setminus \bigcup_{k=1}^n V_k$. Then $V_{n+1} \subset \bigcup_{k=1}^n W_k \setminus V_k$ and so $m_p(V_{n+1}) = 0$. Now $\{V_1, V_2, \ldots, V_{n+1}\}$ is an $\mathcal{R}$-partition of X and, for $0 \leq k \leq n+1$, we have $|f(x) - f(y)| \cdot m_p(V_k) < \epsilon$, if $x, y \in A_k$, which proves that f is m-integrable by Theorem 2.1.

Definition 2.7 *Let $m \in M(\mathcal{R}, E)$ and $f \in \mathbb{K}^X$. We say that f is m-integrable over a measurable set A if $f \cdot \chi_A$ is m-integrable over X. In this case we define*

$$\int_A f\, dm = \int f \chi_A\, dm.$$

If f is m-integrable, then f is $\bar{m}$-integrable. Also χ_A is $\bar{m}$-integrable and so $f\chi_A$ is $\bar{m}$-integrable over X (by [7, Theorem 4.3), which implies that $f\chi_A$ is m-integable. Moreover

$$\int_A f\, dm = \int f\, \chi_A dm = \int f \chi_A\, d\bar{m} = \int_A f\, d\bar{m}.$$

Theorem 2.8 *Let $m \in M(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be m-integrable. Then, given $\epsilon > 0$, there exists $\delta > 0$ such that $p\left(\int_A f\, dm\right) < \epsilon$ for each $A \in \mathcal{R}_m$ with $\bar{m}_p(A) < \delta$.*

Proof : Since f is m-integrable, there exists $W \in \mathcal{R}$ such that $m_p(W^c) = 0$ and $\|f\|_W < d < \infty$. Let $\delta = \epsilon/d$ and let $A \in \mathcal{R}_m$ with $\bar{m}_p(A) < \delta$. Then

$$p\left(\int_A f\, dm\right) = p\left(\int_A f\, d\bar{m}\right) = p\left(\int_{A\cap W} f\, d\bar{m}\right) \leq \|f\|_{A\cap W} \cdot \bar{m}_p(A \cap W) < \epsilon.$$

Theorem 2.9 *Let $m \in M_\tau(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$. Then f is m-integrable iff*

1. *f is $\tau_{\mathcal{R}}$-continuous at every point of the set*

$$G = \bigcup_{p \in cs(E)} \{x : N_{m,p}(x) \neq 0\}.$$

2. *For every $p \in cs(E)$, there exists $W \in \mathcal{R}$, with $m_p(W^c) = 0$ and $\|f\|_W < \infty$.*

Proof : The necessity follows from [7, Theorem 4.2].
Conversely, suppose that (1) and (2) hold and let $p \in cs(E)$ and $\epsilon > 0$. Let $W \in \mathcal{R}$ be such that $m_p(W^c) = 0$ and $\|f\|_W < d < \infty$. Let $\epsilon_1 > 0$ be such that $\epsilon_1 d < \epsilon$ and $\epsilon_1 \cdot \|m\|_p < \epsilon$. The set $Y = \{x : N_{m,p}(x) \geq \epsilon_1\}$ is $\tau_{\mathcal{R}}$-compact (by [7, Theorem 2.6]) and it is contained in W. By (2), f is $\tau_{\mathcal{R}}$- continuous at every point of Y. Hence, for each $x \in Y$, there exists V_x in $\mathcal{R}$ contained in W such that

$$x \in V_x \subset \{y : |f(y) - f(x)| < \epsilon_1\}.$$

By the compactness of Y, Y is covered by a finite number of the V_x, $x \in Y$. Thus, there are pairwise disjoint members $A_1, A_2, \ldots, A_n$ of $\mathcal{R}$ which cover Y such that $A_k \subset W$ and each A_k is contained in some V_x. Let $A_{n+1} = W \backslash \bigcup_1^n A_k$, $A_{n+2} = W^c$. Then

$$m_p(A_{n+1}) = \sup_{x \in A_{n+1}} N_{m,p}(x) \leq \epsilon_1$$

(by [7, Corollary 2.3]) and so

$$|f(x) - f(y)| \cdot m_p(A_{n+1}) \leq d\epsilon_1 < \epsilon$$

if $x, y \in A_{n+1}$. If $x, y \in A_k$, for some $k \leq n$, then

$$|f(x) - f(y)| \cdot m_p(A_k) \leq \epsilon_1 \cdot m_p(A_k) < \epsilon.$$

Now the result follows by Theorem 2.1.

Theorem 2.10 *If $f = g$ m-a.e and g is m-integrable, then f is m-integrable and*

$$\int f\, dm = \int g\, dm.$$

Proof : We will show that f is $\bar{m}$-integrable . The set $A = \{x : f(x) \neq g(x)\}$ is m-negligible and hence $A \in \mathcal{R}_m$. Since g is m-integrable, given $\epsilon > 0$ and $p \in cs(E)$, there exists an $\mathcal{R}$-partition $\{A_1, A_2, \ldots, A_n\}$ of X such that $|g(x) - g(y)| \cdot m_p(A_k) < \epsilon$ if $x, y \in A_k$. If now $\{B_1, B_2, \ldots, B_N\}$ is any $\mathcal{R}_m$-partition of X which is a refinement of each of the partitions $\{A_1, A_2, \ldots, A_n\}$

and $\{A, A^c\}$, then $|f(x) - f(y)| \cdot \bar{m}_p(B_k) < \epsilon$ if $x, y \in B_k$. Indeed this clearly holds if $B_k \subset A$. If $B_k \subset A^c$, then

$$|f(x) - f(y)| \cdot \bar{m}_p(B_k) = |g(x) - g(y)| \cdot \bar{m}_p(B_k) < \epsilon$$

since each B_k is contained in some A_j. This (in view of Theorem 2.1) implies that f is $\bar{m}$-integrable and hence m-integrable. By the same Theorem, if $x_k \in B_k$, then

$$p\left(\int f\, d\bar{m} - \sum_{k=1}^{N} f(x_k)\bar{m}(B_k)\right) \le \epsilon \quad \text{and} \quad p\left(\int g\, d\bar{m} - \sum_{k=1}^{N} g(x_k)\bar{m}(B_k)\right) \le \epsilon.$$

Since, for $B_k \subset A$, we have that $\bar{m}(B_k = 0$ and $f(x_k) = g(x_k)$ when $B_k \subset A^c$, it follows that

$$p\left(\int f\, d\bar{m} - \int g\, d\bar{m}\right) \le \epsilon.$$

This, being true for all $\epsilon > 0$ and all $p \in cs(E)$, implies that

$$\int f\, dm = \int f\, d\bar{m} = \int g\, d\bar{m} = \int g\, dm,$$

which completes the proof.

Theorem 2.11 *Let $m \in M_\sigma(\mathcal{R}, E)$ and suppose that $\mathcal{R}$ is a σ-algebra. If (A_n) is a sequence in $\mathcal{R}$, then for each $p \in cs(E)$ we have*

$$m_p(\liminf A_n) \le \liminf m_p(A_n) \le \limsup m_p(A_n) \le m_p(\limsup A_n).$$

Proof : Let $B_n = \bigcap_{k=n}^{\infty} A_k, \quad G_n = \bigcup_{k=n}^{\infty} A_k$. Then

$$\liminf A_n = \bigcup B_n \quad \text{and} \quad \limsup A_n = \bigcap G_n.$$

Since m is σ-additive, we have $m_p(\liminf A_n) = \sup_n m_p(B_n)$. But

$$m_p(B_n) \le \inf_{k \ge n} m_p(A_k) \le \liminf m_p(A_n).$$

Thus

$$m_p(\liminf A_n) \le \liminf m_p(A_n).$$

Analogously we prove that

$$\limsup m_p(A_n) \le m_p(\limsup A_n)$$

and hence the result follows.

Corollary 2.12 *Let $m \in M_\sigma(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let (A_n) be a sequence in $\mathcal{R}$ such that*

$$\liminf A_n = \limsup A_n = A.$$

Then, for each $p \in cs(E)$, we have that $m_p(A_n) \to m_p(A)$.

Theorem 2.13 *Let $m \in M(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be m-integrable. If $p \in cs(E)$ $\alpha > 0$ ad $\epsilon > 0$, then there exists $g \in S(\mathcal{R})$ such that*

$$m_p^*(\{x : |f(x) - g(x)| \geq \alpha\}) \leq \epsilon.$$

Proof : Since f is m-integrable, there exists an $\mathcal{R}$-partition $\{A_1, A_2, \ldots, A_n\}$ of X such that $|f(x) - f(y)| \cdot m_p(A_k) \leq \epsilon\alpha$ if $x, y \in A_k$. Let $x_k \in A_k$, $g = \sum_{k=1}^n f(x_k)\chi_{A_k}$ and $G = \{x : |f(x) - g(x) \geq \alpha\}$. If $x \in G \cap A_k$, then

$$\epsilon\alpha \geq |f(x) - f(x_k)| \cdot m_p(A_k) \geq \alpha \cdot m_p(A_k)$$

and thus $m_p(A_k) \leq \epsilon$. The set

$$W = \bigcup\{A_k : A_k \cap G \neq \emptyset\}$$

contains G and so $m_p^*(G) \leq m_p(W) \leq \epsilon$.

Theorem 2.14 *Let $m \in M(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be m-integrable. Then, for each $\alpha > 0$, the sets*

$$A_1 = \{x : |f(x)| \geq \alpha\}, \quad A_2 = \{x : |f(x)| > \alpha\}, \quad A_3 = \{x : |f(x)| \leq \alpha\}$$

$$A_4 = \{x : |f(x)| < \alpha\} \quad and \quad A_5 = \{x : |f(x)| = \alpha\}$$

are m-measurable.

Proof : Let $p \in cs(E)$ and $\epsilon > 0$. By the preceding Theorem, there exists $W \in \mathcal{R}$ and $g \in S(\mathcal{R})$ such that $m_p(W) < \epsilon$ and $\{x : |f(x) - g(x)| \geq \alpha\} \subset W$. Let $g = \sum_{k=1}^n \lambda_k \chi_{B_k}$, where $B_1, \ldots, B_n$ are disjoint members of $\mathcal{R}$. Let $B = \{B_k : |\lambda_k| \geq \alpha\}$. Then

$$B \cap W^c \subset \{x : |f(x)| \geq \alpha\} \subset W \cup B.$$

Indeed, let $x \in B \cap W^c$ and assume that $|f(x)| < \alpha$. Since $x \in B$, we have $|g(x)| \geq \alpha$ and so $|g(x) - f(x)| = |g(x)| \geq \alpha$, a contradiction. Hence $B \cap W^c \subset A_1$. Also, if $y \notin W \cup B$, then $|f(y) - g(y)| < \alpha$ and $|g(y)| < \alpha$, which implies that $|f(y)| < \alpha$. Thus $A_1 \subset B \cup W$. Moreover $(W \cup B) \setminus (B \cap W^c) = W$ and $m_p(W) < \epsilon$. This proves that A_1 is m-measurable. In an analogous way we prove that A_2 is measurable. Finally the sets $A_3 = A_2^c$, $A_4 = A_1^c$, and $A_5 = A_1 \setminus A_2$ are measurable.

3 Measurable Functions

Definition 3.1 *If $m \in M(\mathcal{R}, E)$, then a function $f \in \mathbb{K}^X$ is said to be m-measurable, or just measurable if no confusion is possible to arise, if $f^{-1}(A) \in \mathcal{R}_m$ for each clopen subset A of $\mathbb{K}$.*

We have the following two easily verified Lemmas.

Lemma 3.2 *A subset A of X is measurable iff χ_A is measurable.*

Lemma 3.3 *Let A be a closed subset of $\mathbb{K}$ and let*

$$\omega_A : \mathbb{K} \to \mathbb{R}, \quad \omega_A(x) = \inf_{y \in A} |x - y|.$$

Then :

1. *For $x, y \in \mathbb{K}$, we have $\omega_A(x) \le \max\{|x - y|, \omega_A(y)\}$.*
2. *For each $\alpha > 0$, the sets*

   $$\{x : \omega_A(x) \le \alpha\}, \quad \{x : \omega_A(x) < \alpha\} \quad \{x : \omega_A(x) \ge \alpha\}, \quad \{x : \omega_A(x) > \alpha\}$$

   are clopen.

Theorem 3.4 *Let $m \in M(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let $f \in \mathbb{K}^X$. The following are equivalent :*

1. *For each Borel subset B of $\mathbb{K}$, the set $f^{-1}(B)$ is measurable.*
2. *$f^{-1}(A)$ is measurable for each closed subset A of $\mathbb{K}$.*
3. *$f^{-1}(A)$ is measurable for each open subset A of $\mathbb{K}$.*
4. *f is measurable.*

Proof : It is clear that (2) is equivalent to (3) and that (1) $\Rightarrow$ (2) $\Rightarrow$ (4). Also, (3) $\Rightarrow$ (1) since the family of all subsets A of $\mathbb{K}$ for which $f^{-1}(A) \in \mathcal{R}_m$ is a σ-algebra because $\mathcal{R}_m$ is a σ-algebra. Finally, (4) implies (2). Indeed assume that f is measurable and let A be a closed subset of $\mathbb{K}$. Let ω_A be as in the preceding Lemma. Since A is closed, we have that $A = \{s \in \mathbb{K} : \omega_A(s) = 0\}$. Let $A_n = \{s : \omega_A(s) \le 1/n\}$. Each A_n is clopen and thus $B_n = f^{-1}(A_n)$ is measurable. Since $f^{-1}(A) = \bigcap B_n$, the result clearly follows.

Theorem 3.5 *Let $m \in M(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be m-measurable. Then :*

1. *If $\phi : \mathbb{K} \to \mathbb{K}$ is continuous, then the function $\phi \circ f$ is measurable.*
2. *For each $g \in S(\mathcal{R}_m)$, the functions $h_1 = gf$ and $h_2 = g + f$ are measurable.*

Proof : 1). It follows from the fact that $\phi^{-1}(A)$ is clopen in $\mathbb{K}$ for each clopen A.
2). There exists an $\mathcal{R}_m$-partition $\{A_1, \ldots, A_n\}$ of X, and λ_k in $\mathbb{K}$ such that $g = \sum_{k=1}^{n} \lambda_k \chi_{A_k}$, $\lambda_n = 0$, $\lambda_k \neq 0$ for $k < n$ (we may have $A_n = \emptyset$). Now, for A clopen subset of $\mathbb{K}$, we have

$$h_1^{-1}(A) = \bigcup_{k=1}^{n} h_1^{-1}(A) \cap A_k.$$

If $k < n$, then

$$h_1^{-1}(A) \cap A_k = A_k \bigcap \left[f^{-1}(\lambda_k^{-1} A)\right].$$

Also

$$h_1^{-1}(A) \cap A_n \in \{A_n, \emptyset\}.$$

Hence each $h_1^{-1}(A) \cap A_k$ is measurable and so $h_1^{-1}(A)$ is measurable, which proves that h_1 is measurable. To prove that h_2 is measurable, it suffices to show that, for $G \in \mathcal{R}_m$ and $\lambda \in \mathbb{K}$, the function $h = f + \lambda \chi_G$ is measurable. For such an h and A clopen subset of $\mathbb{K}$, we have

$$h^{-1}(A) = \left[G \cap f^{-1}(-\lambda + A)\right] \bigcup \left[G^c \cap f^{-1}(A)\right],$$

and the result follows.

Theorem 3.6 *Let $m \in M_\tau(\mathcal{R}, E)$. Then :*

1. *An $f \in \mathbb{K}^X$ is measurable iff it is $\tau_{\mathcal{R}_m}$-continuous.*

2. *If f, g are measurable, then $f + g$ and fg are measurable.*

Proof : 1). It follows from the fact that, when m is τ-additive, a subset of X is in $\mathcal{R}_m$ iff it is $\tau_{\mathcal{R}_m}$-clopen.
2). It is a consequence of (1) since the sum and the product of two continuous functions are continuous.

Theorem 3.7 *Let $m \in M(\mathcal{R}, E)$ and let $f, g \in \mathbb{K}^X$ with $f = g$ m-a.e. If g is measurable, then f also is measurable.*

Proof : The set $G = \{x : f(x) \neq g(x)\}$ is negligible and hence measurable. For A a clopen subset of $\mathbb{K}$, we have

$$f^{-1}(A) = \left[f^{-1}(A) \cap G\right] \bigcup \left[f^{-1}(A) \cap G^c\right] = \left[f^{-1}(A) \cap G\right] \bigcup \left[g^{-1}(A) \cap G^c\right].$$

Since $f^{-1}(A) \cap G$ is negligible and hence measurable, the result follows.

Theorem 3.8 *Let $m \in M(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra. If f, g are measurable functions and $\lambda \in \mathbb{K}$, then the sets*

$$G_1 = \{x : |f(x)| > |g(x)|\}, \quad G_2 = \{x : |f(x)| \geq |g(x)|\},$$

$$G_3 = \{x : |f(x)| = |g(x)|\}, \quad G_4 = \{x : f(x) = \lambda\}$$

are measurable.

Proof : For each rational number r, the set

$$F_r = \{x : |f(x)| > r\} \bigcap \{x : x : |g(x)| < r\}$$

is measurable. Since $\mathcal{R}$ is a σ-algebra, $\mathcal{R}_m$ is also a σ-algebra and thus the set

$$G_1 = \bigcup \{F_r : r > 0, \quad r \quad rational\}$$

is measurable. Analogously the set $B = \{x : |g(x)| > |f(x)|\}$ is measurable and so $G_2 = B^c$ is measurable. Also $G_3 = G_2 \setminus G_1$ is measurable. Finally the function $h = f - \lambda$ is measurable, by Theorem 3.5, and so the set

$$G_4 = \bigcap_{n=1}^{\infty} \{x : |h(x)| < 1/n\}$$

is measurable.

Theorem 3.9 *Let $m \in M(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be measurable. Then f is $\tau_{\mathcal{R}_m}$-continuous at every point of the set*

$$Z = \bigcup_{p \in cs(E)} \{x : N_{m,p}(x) \neq 0\}.$$

Proof: Let $N_{m,p}(x) = d > 0$ and let $\epsilon > 0$. The set $G = \{x : |f(y) - f(x)| \leq \epsilon\}$ is measurable. Hence, there are $V, W \in \mathcal{R}$ such that $V \subset G \subset W$ and $m_p(W \setminus V) < d$. Since $x \in W$ and $N_{m,p}(x) > m_p(W \setminus V)$, it follows that $x \in V \subset G$, which proves that f is continuous at x.

Corollary 3.10 *Let $m \in M_\tau(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be measurable. If there exists an integrable function g such that $|f| \leq |g|$, then f is integrable.*

Proof : Given $p \in cs(E)$, there exists $W \in \mathcal{R}$ such that $\|g\|_W < \infty$ and $m_p(W^c) = 0$. By the preceding Theorem and the Theorem 2.9, f is $\bar{m}$-integrable and so f is m-integrable.

Theorem 3.11 *Let $m \in M(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let (f_n) be a sequence of measurable functions which converges to some f m-almost everywhere. Then f is measurable.*

Proof : Let A be a clopen subset of $\mathbb{K}$ and let $B_n = f_n^{-1}(A)$. The set $B = \liminf B_n$ is in $\mathcal{R}_m$ since $\mathcal{R}_m$ since is a σ-algebra. Let $Z = \{x : f(x) = \lim f_n(x)\}$. Then Z^c is m-negligible and hence measurable. Moreover, $f^{-1}(A) \cap Z = B \cap Z$. Indeed, let $x \in f^{-1}(A) \cap Z$. Since $\lim f_n(x) = f(x) \in A$, there exists a k such that
$x \in \bigcap_{n \geq k} B_n \subset B$. Conversely, if $x \in B \cap Z$, then there exists a k such that $x \in \bigcap_{n \geq k} B_n$, and so $f_n(x) \in A$ for all $n \geq k$. Since A is closed and $f_n(x) \to f(x)$, it follows that $f(x) \in A$ and so $x \in f^{-1}(A) \cap Z$. Now $B \cap Z$ is measurable and

$$f^{-1}(A) = [B \cap Z] \bigcup [f^{-1}(A) \cap Z^c].$$

As $f^{-1}(A) \cap Z^c$ is negligible, it is measurable and so $f^{-1}(A)$ is measurable. Hence the result follows.

Theorem 3.12 (**Egoroff's Theorem**) *Let $m \in M_\tau(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let (f_n) be a sequence of measurable functions which converges m-a.e to some f. Then for each $\epsilon > 0$ and each $p \in cs(E)$, there exists $A \in \mathcal{R}$, with $m_p(A^c) < \epsilon$, such that $f_n \to f$ uniformly on A.*

Proof : Let G be an m-negligible set such that $f_n(x) \to f(x)$ for all $x \in G^c$ and let $p \in cs(E)$ and $\epsilon > 0$. By the preceding Theorem, f is measurable.
Claim. For each $\delta > 0$, there exist $B \in \mathcal{R}$, with $m_p(B^c) \leq \epsilon$, and an integer N such that $|f_n(x) - f(x)| < \delta$ for all $x \in B$ and all $n \geq N$. In fact, let

$$A_n = \{x \in X : |f_n(x) - f(x)| \geq \delta\} \bigcap G^c \quad \text{and} \quad D_N = \bigcup_{n \geq N} A_n.$$

Since m is τ-additive, each $f_n - f$ is measurable (by Theorem 3.4) and so A_n is measurable, which implies that D_N is measurable since $\mathcal{R}$ is a σ-algebra. Moreover $D_N \downarrow \emptyset$ since $f_n(x) \to f(x)$ for all $x \in G^c$. As $\bar{m}$ is σ-additive, there exists an N such that $\bar{m}_p(D_N \cup G) = \bar{m}_p(D_N) < \epsilon$. There are $V, W \in \mathcal{R}$ such that $V \subset D_N \cup G \subset W$ and $m_p(W \setminus V) < \epsilon$. Now

$$m_p(W) = \max\{m_p(V), m_p(W \setminus V)\} \leq \max\{\bar{m}_p(D_N \cup G), \epsilon\} = \epsilon.$$

Taking $B = W^c$, we see that if $x \in B$, then $x \notin D_N \cup G$ and so $x \notin A_n$, for each $n \geq N$, i.e $|f_n(x) - f(x)| < \delta$. Thus the claim follows.
By our claim, there are $n_1 < n_2 < \ldots$, and sets $B_k \in \mathcal{R}$, with $m_p(B_k) < \epsilon$ and $|f_n - f(x)| < 1/k$ for all $x \notin B_k$ and all $n \geq n_k$. For $A = \bigcup B_k$, we have that $m_p(A) = \sup_k m_p(B_k) \leq \epsilon$. Moreover, $f_n \to f$ uniformly on A^c. In fact, given $\delta > 0$, choose $k > 1/\delta$. If $x \in A^c \subset B_k^c$, we have $|f_n(x) - f(x)| \leq 1/k < \delta$ for all $n \geq n_k$. This completes the proof.

Theorem 3.13 *Let $m \in M(\mathcal{R}, E)$, where E is metrizable, and let (f_n) be a sequence in $\mathbb{K}^X$ and $f \in \mathbb{K}^X$. If, for each $p \in cs(E)$ and each $\epsilon > 0$, there exists an A in $\mathcal{R}$, with $m_p(A) < \epsilon$, such that (f_n) converges uniformly to f on A^c, then $f_n(x) \to f(x)$ m-a.e.*

Proof : Let (p_n) be an increasing sequence of continuous seminorms on E such that, for each $p \in cs(E)$, there exists an n with $p \leq p_n$. For each k, there exists $A_k \in \mathcal{R}$, with $m_{p_k}(A_k) < 1/k$, such that $f_n \to f$ uniformly on A_k^c. Let $A = \bigcap A_k$ and let $p \in cs(E)$. Choose k such that $p \leq p_k$. Then, for each $n \geq k$, we have

$$m_p^*(A) \leq m_p(A_n) \leq m_{p_n}(A_n) < 1/n \to 0,$$

and hence A is negligible. Moreover, $f_n(x) \to f(x)$ for all $x \in A^c$.

4 Convergence in Measure

Let $m \in M(\mathcal{R}, E)$.

Definition 4.1 *A net (g_δ) in $\mathbb{K}^X$ converges in measure, with respect to m, to some $f \in \mathbb{K}^X$ if, for each $p \in cs(E)$ and each $\alpha > 0$, we have*

$$\lim_\delta m_p^*(\{x : |g_\delta(x) - f(x)| \geq \alpha\}) = 0.$$

Theorem 4.2 *Let $m \in M_\sigma(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let (f_n) be a sequence in $\mathbb{K}^X$ which converges in measure to both f and g. Then $f = g$ m-a.e.*

Proof : For each positive integer k, let

$$A_{nk} = \{x : |f_n(x) - f(x)| \geq 1/k\}, \quad B_{nk} = \{x : |g(x) - f_n(x)| \geq 1/k\},$$

$$G_k = \{x : |f(x) - g(x)| \geq 1/k\}.$$

Then $G_k \subset A_{nk} \bigcup B_{nk}$ and so

$$m_p^*(G_k) \leq \max\{m_p^*(A_{nk}), \quad m_p^*(B_{nk})\},$$

for all n. It follows that $m_p^*(G_k) = 0$ for all $p \in cs(E)$, and so G_k is negligible. Since m is σ-additive and $\mathcal{R}$ a σ-algebra, it follows that the set

$$G = \{x : f(x) \neq g(x)\} = \bigcup G_k$$

is negligible, and thus $f = g$ m-a.e

Theorem 4.3 *Let $m \in M(\mathcal{R}, E)$ and $f \in \mathbb{K}^X$. Then, f is m-integrable iff*

1. *There exists a net (g_δ) in $S(\mathcal{R})$ which converges in measure to f.*

2. *For each $p \in cs(E)$ there exists a $W \in \mathcal{R}$, with $m_p(W^c) = 0$, such that f is bounded on W.*

Proof : Assume that f is integrable. Then (2) holds by Theorem 2.1. To prove (1), we consider the set $\Delta = \{(n,p) : n \in \mathbb{N}, p \in cs(E)\}$. We make Δ into a directed set by defining $(n_1, p_1 \geq (n_2, p_2)$ iff $n_1 \geq n_2$ and $p_1 \geq p_2$.
Claim: For each $\delta = (n,p)$, there exist $h_\delta \in S(\mathcal{R})$ and $G_\delta \in \mathcal{R}$ such that

$$m_p(G_\delta) < 1/n \quad \textit{and} \quad A_\delta = \{x : |h_\delta(x) - f(x)| \geq 1/n\} \subset G_\delta.$$

Moreover, we can choose h_δ so that $h_\delta(X) \subset f(X)$.
Indeed, there exists an $\mathcal{R}$-partition $\{B_1, \ldots, B_N\}$ of X such that , for each $1 \leq k \leq N$, we have $|f(x) - f(y)| \cdot m_p(B_k) < 1/n^2$ if $x, y \in B_k$. Choose $x_k \in B_k$ and set $g_\delta = \sum_{k=1}^N f(x_k)\chi_{B_k}$. Let

$$A_\delta = \{x : |h_\delta(x) - f(x)| \geq 1/n\} \quad \textit{and} \quad G_\delta = \bigcup\{B_k : B_k \cap A_\delta \neq \emptyset\}.$$

If $x \in B_k \cap A_\delta$, then

$$1/n^2 > |f(x) - f(x_k)| \cdot m_p(B_k) \geq 1/n \cdot m_p(B_k),$$

and so $m_p(B_k) < 1/n$. It follows that $m_p(G_\delta) < 1/n$ and clearly $A_\delta \subset G_\delta$. This proves the claim. Now $h_\delta \to f$ in measure. In fact, let $p_o \in cs(E)$, $\alpha > 0$ and $\epsilon > 0$. Choose $n_o > 1/\alpha, 1/\epsilon$. For $\delta = (n,p) \geq \delta_o = (n_o, p_o)$, let

$$Z_\delta = \{x : |g_\delta(x) - f(x)| \geq \alpha\}.$$

Then $Z_\delta \subset A_\delta \subset G_\delta$ and so $m_p^*(Z_\delta) \leq m_p(G_\delta) < 1/n < \epsilon$. This proves that $h_\delta \to f$ in measure.
Conversely, suppose that (1) and (2) hold and let $p \in cs(E)$ and $\epsilon > 0$. By (2), there exists $W \in \mathcal{R}$, with $m_p(W^c) = 0$, such that $\|f\|_W < d < \infty$. Let (g_δ) be a net in $S(\mathcal{R})$ which converges in measure to f. Choose $\alpha > 0$ such that $\alpha \cdot m_p(X) < \epsilon$. There exists a δ_o such that $m_p^*(Z_{\delta_o}) < \epsilon/d$, where

$$Z_{\delta_o} = \{x : |g_{\delta_o}(x) - f(x)| \geq \alpha\}.$$

There exist an $\mathcal{R}$-partition $\{W_1, \ldots, W_N\}$ of X and $\lambda_i \in \mathbb{K}$ such that $g_{\delta_o} = \sum_{i=1}^N \lambda_i \chi_{W_i}$. There is a $V \in \mathcal{R}$ containing Z_{δ_o} such that $m_p(V) < \epsilon/d$. Let $\{V_1, \ldots, V_n\}$ be any $\mathcal{R}$-partition of X, which is a refinement of each of the partitions $\{W_1, \ldots, W_N\}$, $\{W, W^c\}$, and $\{V, V^c\}$. Let $1 \leq i \leq n$ and $x, y \in V_i$. We will prove that

$$|f(x) - f(y)| \cdot m_p(V_i) \leq \epsilon.$$

This is clearly true if $V_i \subset W^c$. So, assume that $V_i \subset W$. If $V_i \subset V$, then

$$|f(x) - f(y)| \cdot m_p(V_i) \leq d \cdot m_p(V) \leq \epsilon.$$

Finally, if $V_i \subset V^c$, then (since $g_{\delta_o}(x) = g_{\delta_o}(y)$ as x, y are in some W_j) we have

$$|f(x) - f(y)| \leq \max\{|f(x) - g_{\delta_o}(x)|, \quad |g_{\delta_o}(y) - f(y)|\} < \alpha$$

and so

$$|f(x) - f(y)| \cdot m_p(V_i) \leq \alpha \cdot m_p(X) < \epsilon.$$

Now the result follows from Theorem 2.1.

Theorem 4.4 *Let $m \in M(\mathcal{R}, E)$ and let $(g_\delta)_{\delta \in \Delta}$ be a net in $\mathbb{K}^X$ which converges in measure to some f. If E is metrizable, then there exist $\delta_1 \leq \delta_2 \leq \ldots$ such that the sequence (g_{δ_n}) converges in measure to f.*

Proof : Let (p_n) be an increasing sequence of continuous seminorms on E such that, for each $p \in cs(E)$, there exists an n with $p \leq p_n$. There is an increasing sequence (δ_n) in Δ such that

$$m^*_{p_n}(\{x : |g_\delta(x) - f(x)| \geq 1/n\}) < 1/n$$

for all $\delta \geq \delta_n$. Let $h_n = g_{\delta_n}$. Then $h_n \to f$ in measure. Indeed, let $p \in cs(E)$, $\alpha > 0$ and $\epsilon > 0$. Choose $n_o > 1/\alpha, 1/\epsilon$ with $p_{n_o} \geq p$. Then, for $n \geq n_o$, we have

$$\begin{aligned} m^*_p(\{x : |h_n(x) - f(x)| \geq \alpha\}) &\leq m^*_p(\{x : |h_n(x) - f(x)| \geq 1/n\}) \\ &\leq m^*_{p_n}(\{x : |h_n(x) - f(x)| \geq 1/n\}) < 1/n < \epsilon. \end{aligned}$$

Thus $h_n \to f$ in measure and the result follows.

Corollary 4.5 *If $f \in \mathbb{K}^X$ is m-integrable and E metrizable, then there exists a sequence (g_n) in $S(\mathcal{R})$ which converges in measure to f. Moreover, we can choose (g_n) so that $g_n(X) \subset f(X)$ for all n.*

Theorem 4.6 *Let $m \in M_\sigma(\mathcal{R}, E)$, where E is metrizable, and consider on X the topology $\tau_\mathcal{R}$. Let (f_n) be a sequence in $\mathbb{K}^X$ which converges in measure to some f. Then, there exist a subsequence (f_{n_k}) and an F_σ set F such that F is a support set for m and $f_{n_k} \to f$ pointwise on F. If $\mathcal{R}$ is a σ-algebra, then we may choose F to be in $\mathcal{R}$.*

Proof : Let (p_n) be an increasing sequence of continuous seminorms on E such that, for each $p \in cs(E)$, there exists an n with $p \leq p_n$. Choose inductively $n_1 < n_2 < \ldots$ such that

$$m^*_{p_k}(\{x : |f_n(x) - f(x)| \geq 1/k\}) < 1/k$$

for all $n \geq n_k$. Let

$$A_k = \{x : |f_n(x) - f(x)| \geq 1/k\}$$

and let $V_k \in \mathcal{R}$, containing A_k, such that $m_{p_k}(V_k) < 1/k$. Set

$$A = \bigcap_{N=1}^{\infty} \bigcup_{k \geq N} V_k, \quad F = X \setminus A.$$

Then F is an F_σ set and $F \in \mathcal{R}$ if $\mathcal{R}$ is a σ-algebra. If $V \in \mathcal{R}$ is contained in A, then $p_k(m(V)) = 0$ for all k. Indeed, for all N, we have $V \subset \bigcup_{i \geq N} V_i$. So, if $N > k$, then

$$m_{p_k}(V) \leq \sup_{i \geq N} m_{p_k}(V_i) \leq \sup_{i \geq N} m_{p_i}(V_i) \leq 1/N$$

and so $m_{p_k}(V) = 0$. This proves that F is a support set for m. Finally, let $x \in F$ and let N_o be such that $x \notin \bigcup_{i \geq N_o} V_i$. For $k \geq N_o$, we have $x \notin V_k$ and so $|f_{n_k}(x) - f(x)| < 1/k \to 0$. This clearly completes the proof.

Theorem 4.7 *Let $m \in M_\sigma(\mathcal{R}, E)$, where E is metrizable and $\mathcal{R}$ a σ-algebra. If f is m-integrable, then f is m-measurable.*

Proof : By Corollary 4.5, there exists a sequence (g_n) in $S(\mathcal{R})$ which converges in measure to f. In view of the preceding Theorem, there exist a subsequence (g_{n_k}) and a set $F \in \mathcal{R}$ such that F is a support set for m and $g_{n_k} \to f$ pointwise on F. Since each g_{n_k} is measurable, it follows that f is measurable by Theorem 3.11.

Theorem 4.8 *Let $m \in M_\sigma(\mathcal{R}, E)$, where E is metrizable and $\mathcal{R}$ a σ-algebra. If a sequence (f_n) of measurable functions converges in measure to some f, then f is measurable.*

Proof : By Theorem 4.6 there exist a se subsequence (f_{n_k}) and a set $F \in \mathcal{R}$ such that F is a support set for m and $f_{n_k} \to f$ pointwise on F. Now the result follows from Theorem 3.11.

Theorem 4.9 *Let $m \in M_\sigma(\mathcal{R}, E)$, $p \in cs(E)$ and $\epsilon > 0$. Then :*

1. *If $f \in \mathbb{K}^X$ is measurable, then there exists a $d > 0$ such that*

$$m_p^*(\{x : |f(x)| > d\}) < \epsilon.$$

2. *If (g_n) is a sequence of measurable functions which converges in measure to some g, then there exists $\alpha > 0$ such that $m_p^*(\{x : |g(x)| > \alpha\}) < \epsilon$.*

Proof : 1). Let $V_n = \{x : |f(x)| > n\}$. Then $V_n \in \mathcal{R}_m$ and $V_n \downarrow \emptyset$. Since $\bar{m}$ is σ-additive, there exists an n such that $\bar{m}_p^*(V_n) < \epsilon$.
2). Let $A_n = \{x : |g_n(x) - g(x)| \geq 1\}$. There exists an n such that $m_p^*(A_n) < \epsilon$. By (1), there exists $\alpha > 1$ sauch that, if $B = \{x : |g_n(x)| > \alpha\}$, then $m_p^*(B) < \epsilon$. If $A = \{x : |g(x)| > \alpha\}$, then $A \subset B \cup A_n$ and so

$$m_p^*(A) \leq \max\{m_p^*(B), m_p^*(A_n)\} < \epsilon.$$

Theorem 4.10 *Let $m \in M_\sigma(\mathcal{R}, E)$ and let (f_n) and (g_n) be two sequences of measurable functions which converge in measure to f, g, respectively. Then $f_n + g_n \to f + g$ and $f_n g_n \to fg$ in measure.*

Proof : It is easy to see that $(f_n + g_n)$ converges in measure to $f + g$. To prove that the sequence $(f_n g_n)$ converges in measure to fg, we first prove that $f_n g \to fg$ in measure. Indeed, let $p \in cs(E)$, $\alpha > 0$ and $\epsilon > 0$. By the preceding Theorem, there exists a $d > 0$ such that, if $A = \{x : |g(x)| > d\}$, then $m_p^*(A) < \epsilon$. Let

$$A_n = \{x : |f_n(x)g(x)(-f(x)g(x)| \geq \alpha\}, \quad B_n = \{x : |f_n(x) - f(x)| \geq \alpha/d\}.$$

Then $A_n \subset B_n \cup A$. There exists an n_o such that $m_p^*(B_n) < \epsilon$ for $n \geq n_o$. Thus, for $n \geq n_o$, we have

$$m_p^*(A_n) \leq \max\{m_p^*(B_n), m_p^*(A)\} < \epsilon,$$

which proves our claim.
Next we show that $f_n^2 \to f^2$ (and analogously $g_n^2 \to g^2$) in measure. Indeed let $h_n = f_n - f$. Then $h_n \to 0$ in measure. Since, for $\alpha > 0$, we have

$$\{x : |h_n^2(x)| \geq \alpha\} = \{x : |h_n(x)| \geq \alpha^{1/2}\},$$

it follows that $h_n^2 \to 0$ in measure. Now $f_n^2 - f^2 = h_n^2 + 2(f_n f - f^2) \to 0$ in measure and so $f_n^2 \to f^2$ in measure.
Next we observe that

$$(f_n + g_n)(f + g) = f_n f + g_n f + f_n g + g_n g \to f^2 + 2fg + g^2$$

in measure. If $\phi_n = (f_n + g_n) - (f + g)$, then $\phi_n \to 0$ in measure and so $\phi_n^2 \to 0$ in measure. Now

$$(f_n + g_n)^2 - (f + g)^2 = \phi_n^2 + 2\left[(f_n + g_n)(f + g) - (f + g)^2\right] \to 0$$

in measure. Finally,

$$f_n g_n = \frac{1}{2}\left[(f_n + g_n)^2 - f_n^2 - g_n^2\right] \to \frac{1}{2}\left[(f + g)^2 - f^2 - g^2\right] = fg$$

in measure. Hence the result follows.

Theorem 4.11 *Let $m \in M_\sigma(\mathcal{R}, E)$, where E is metrizable and $\mathcal{R}$ a σ-algebra. Let $f, g \in \mathbb{K}^X$ be such that f is m-integrable and g m-measurable. Then $f + g$ and gf are m-measurable.*

Proof : By Corollary 4.5, there exists a sequence (h_n) of $\mathcal{R}$-simple functions which converges in measure to f. In view of the preceding Theorem, the sequence $(h_n g)$ converges in measure to fg. Each $h_n g$ is measurable by Theorem 3.5. Hence fg is measurable by Theorem 4.8. The same Theorem implies that $f + g$ is measurable since $h_n + g \to f + g$ in measure and each $h_n + g$ is measurable by Theorem 3.5.

Theorem 4.12 *Let $m \in M_\tau(\mathcal{R}, E)$ and let $(f_\delta)_{\delta \in \Delta}$ be a net in $\mathbb{K}^X$ which converges in measure to some f. Then, there exists a support set F for m and a subnet of (f_δ) which converges to f pointwise on F.*

Proof : Let $\Xi = \{(\delta, p, k) : \delta \in \Delta, p \in cs(E), k \in \mathbb{N}\}$ and make Ξ into a directed set by defining $(\delta, p, k) \geq (\delta_1, p_1, k_1)$ iff $\delta \geq \delta_1$, $p \geq p_1$ and $k \geq k_1$. Let $\xi = (\delta, p, k)$. There exists $\delta_1 = \psi(\xi) \geq \delta$ such that

$$m_p^*(\{x : |f_{\delta_1}(x) - f(x)| \geq 1/k\}) < 1/k.$$

In this way we get a subnet $(f_{\psi(\xi)})_{\xi \in \Xi}$ of (f_δ). Let

$$G_\xi = \{x : |f_{\psi(\xi)}(x) - f(x)| \geq 1/k\}$$

and choose $W_\xi \in \mathcal{R}$ containing G_ξ and such that $m_p(W_\xi) < 1/k$. Let

$$A = \bigcap_{\xi \in \Xi} \bigcup_{\xi' \geq \xi} W_{\xi'}, \quad F = X \setminus A.$$

Then : 1. $f_{\psi(\xi)}(x) \to f(x)$ for all $x \in F$. In fact, let $x \in F$. There exists a $\xi_1 = (\delta_1, p_1, k_1)$ such that Now, for $\xi = (\delta, p, k) \geq \xi_1$, we have

$$|f_{\psi(\xi)}(x) - f(x)| < 1/k \to 0 \quad as \quad k \to \infty.$$

Thus $f_{\psi(\xi)}(x) \to f(x)$.
2. F is a support set for m. Indeed, Let $W \in \mathcal{R}$ be contained in A and let $\xi_o = (\delta_o, p_o, k_o) \in \Xi$. Then $W \subset \bigcup_{\xi' \geq \xi_o} W_{\xi'}$. Since m is τ-additive, we have

$$m_{p_o}(W) \leq \sup_{\xi' \geq \xi_o} m_{p_o}(W_{\xi'}).$$

But, for $\xi' = (\delta, p, k) \geq \xi_o$, we have

$$m_{p_o}(W_{\xi'}) \leq m_p(W_{\xi'}) < 1/k \leq 1/k_o.$$

It follows that $m_{p_o}(W) = 0$ for all $p_o \in cs(E)$, which proves that F is a support set for m. This completes the proof.

Theorem 4.13 *(***Dominated Convergence Theorem***) Let $m \in M_\tau(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra and E metrizable, and let (f_n) be a sequence of integrable functions which converges m-a.e to some f. If there exists an integrable function g such that $|f_n| \leq |g|$ for all n, then f is integrable and*

$$\int f\, dm = \lim \int f_n\, dm.$$

Proof : Let $p \in cs(E)$ and $\epsilon > 0$. Since g is integrable, there exists a $W \in \mathcal{R}$ such that $m_p(W^c) = 0$ and $\|g\|_W < d < \infty$. Each f_n is measurable by Theorem 4.7. By Egoroff's Theorem, there exists $A \in \mathcal{R}$, with $m_p(A^c) < \epsilon/d$, such that $f_n \to f$ uniformly on A. Also, there exists an m-negligible set B such that $f_n(x) \to f(x)$ for all $x \in B^c$. Clearly $|f| \leq |g|$ on B^c. For each k, there exists

$B_k \in \mathcal{R}$ with $B \subset B_k$ and $m_p(B_k) < 1/k$. The set $F = \bigcap B_k$ is in $\mathcal{R}$ and $m_p(F) = 0$. Since $f_n \to f$ uniformly on A, there exists n_o such that

$$\|f_n - f\|_A < \min\{\epsilon/d, \epsilon/\|m\|_p\}.$$

for all $n \geq n_o$. Let now $n \geq n_o$. Since f_n is integrable, there exists an $\mathcal{R}$-partition $\{A_1, \ldots, A_N\}$ of X, which is a refinement of each of the partitions $\{F, F^c\}$, $\{W, W^c\}$, $\{A, A^c\}$, such that, for all $1 \leq k \leq N$, we have $|f_n(x) - f_n(y)| \cdot m_p(A_k) < \epsilon$ if $x, y \in A_k$. Now, if $x, y \in A_k$, then $|f(x) - f(y)| \cdot m_p(A_k) \leq \epsilon$. In fact, this is clearly true if $A_k \subset W^c$ or $A_k \subset F$. So assume that $A_k \subset F^c \cap W$. Then, for $x, y \in A_k$, we have

$$|f(x) - f(y)) \leq \max\{|f(x) - f_n(x)|, |f_n(x) - f_n(y)|, |f_n(y) - f(y)|\}.$$

It follows from this that $|f(x) - f(y)| \cdot m_p(A_k) \leq \epsilon$. This proves that f is m-integrable. Moreover, if $x_k \in A_k$, then

$$p\left(\int f\,dm - \sum_{k=1}^{N} f(x_k)m(A_k)\right), \quad p\left(\int f_n\,dm - \sum_{k=1}^{N} f_n(x_k)m(A_k)\right) \leq \epsilon.$$

Also, for $1 \leq k \leq N$, we have $|f(x_k) - f_n(x_k)| \cdot p(m(A_k) \leq \epsilon$. Indeed, this is clearly true if $A_K \subset W^c$ or $A_k \subset F$. So assume that $A_k \subset F^c \cap W$. If $A_k \subset A$, then

$$|f(x_k) - f_n(x_k)| \cdot p(m(A_k) \leq \|f - f_n\|_A \cdot \|m\|_p \leq \epsilon,$$

while for $A_k \subset A^c$, we have

$$|f(x_k) - f_n(x_k)| \cdot p(m(A_k \leq d \cdot m_p(A^c) \leq \epsilon.$$

It follows from the above that

$$p\left(\int f\,dm - \int f_n\,dm\right) \leq \epsilon$$

for alll $n \geq n_o$. Thus

$$\int f\,dm = \lim \int f_n\,dm.$$

Theorem 4.14 *Let $m \in M_\tau(\mathcal{R}, E)$, where E is metrizable and $\mathcal{R}$ a σ-algebra, and let $f \in \mathbb{K}^X$. Then, f is m-integrable iff it is measurable (equivalently $\tau_{\mathcal{R}_m}$-continuous) and, for each $p \in cs(E)$, there exists a $W \in \mathcal{R}$ such that $m_p(W^c) = 0$ and f is bounded on W.*

Proof: The necessity follows from Theorems 4.7 and 2.1. Conversely, suppose that the condition is satisfied. We will show that f is $\bar{m}$-integrable and hence m-integrable. Let $p \in cs(E)$, $\epsilon > 0$ and let $W \in \mathcal{R}$ be such that f is bounded on W and $m_p(W^c) = 0$. Let $f_1 = f \cdot \chi_W$. Since f is measurable, it is $\tau_{\mathcal{R}_m}$-continuous (by theorem 3.6) and so f_1 is $\bar{m}$-integrable by [7, Theorem 4.11]. Hence there exists a $\mathcal{R}_m$-partition $\{A_1, \dots, A_n\}$ of X such that, for all $1 \leq k \leq n$, we have $|f_1(x) - f_1(y)| \cdot m_p(A_k) < \epsilon$ if $x, y \in A_k$. Let now $\{B_1, \dots, B_N\}$ be any $\mathcal{R}_m$-partition of X which is a refinement of both $\{A_1, \dots, A_n\}$ and $\{W, W^c\}$. Then, for $1 \leq k \leq N$ and $x, y \in B_k$, we have $|f(x) - f(y)| \cdot m_p(B_k) < \epsilon$. Indeed, this clearly holds if $B_k \subset W^c$. Suppose that $B_k \subset W$. Then $f = f_1$ on B_k and so $|f(x) - f(y)| \cdot m_p(B_k) < \epsilon$ since B_k is contained in some A_i. Now the result follows.

Theorem 4.15 *Let $m \in M_\tau(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let (f_n) be a sequence of measurable functions which converges m-a.e. to some f. Then $f_n \to f$ in measure and f is measurable,*

Proof : Let $p \in cs(E)$, $\alpha > 0$ and $A_n = \{x : |f_n(x) - f(x)| \geq \alpha\}$. Given $\epsilon > 0$, there exists (by Egoroff's Theorem) a set $A \in \mathcal{R}$, with $m_p(A^c) < \epsilon$, such that $f_n \to f$ uniformly on A. Hence, there exists an n_o such that $\|f_n - f\|_A < \alpha$ for all $n \geq n_o$. Now, for $n \geq n_o$, we have $A_n \subset A^c$ and so $m_p^*(A_n) \leq m_p(A^c) < \epsilon$. Hence $f_n \to f$ in measure. Also f is measurable by Theorem 3.11.

Theorem 4.16 *$m \in M_\tau(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be measurable. Then, there exists a net (g_δ) in $S(\mathcal{R})$ which converges in measure to f. In case E is metrizable, there exists a sequence (h_n) in $S(\mathcal{R})$ converging to f in measure.*

Proof : We prove first the following
Claim : For each $\epsilon > 0$ and each $p \in cs(E)$, there exist $A \in \mathcal{R}$, with $m_p(A^c) < \epsilon$, and $g \in S(\mathcal{R})$ such that $\|f - g\|_A \leq \epsilon$. In fact, consider the equivalence relation $\sim$ on X, $x \sim y$ iff $|f(x) - f(y)| \leq \epsilon$. Let $(B_i)_{i \in I}$ be the family of all equivalence classes and let $x_i \in B_i$. Then $B_i = \{x : |f(x) - f(x_i)| \leq \epsilon\}$ and so B_i is measurable since f is measurable. For $J \subset I$ finite, let $G_J = \left(\bigcup_{i \in J} B_i\right)^c$. Then G_J is measurable and $G_J \downarrow \emptyset$. Since $\bar{m}$ is τ-additive, there exists a $J = \{i_1, \dots, i_n\}$ such that $\bar{m}_p(G_J) < \epsilon$. For $1 \leq r \leq n$, there are $V_r, W_r \in \mathcal{R}$ such that $V_r \subset B_{i_r} \subset W_r$ and $m_p(W_r \setminus V_r) < \epsilon$. Let $y_r \in V_r$ and $g = \sum_{r=1}^n f(y_r)\chi_{V_r}$. If $A = \bigcup_{r=1}^n V_r$, then

$$A^c = \bigcap_{r=1}^n V_r^c \subset G_J \bigcup \left(\bigcup_{r=1}^n W_r \setminus V_r \right).$$

Thus,

$$m_p(A^c) = \bar{m}_p(A^c) \leq \max\{\bar{m}_p(G_J), m_p(W_1 \setminus V_1), \dots, m_p(W_n \setminus V_n)\} < \epsilon.$$

Moreover, if $x \in A$, then $x \in V_r$, for some r, and so $|f(x) - g(x)| = |f(x) - f(y_r)| \leq \epsilon$. thus $\|f - g\|_A \leq \epsilon$ and the claim follows.
Let now $\Delta = \{(n, p) : n \in \mathbb{N}, p \in cs(E)\}$. For $\delta = (n, p) \in \Delta$, there exist a

function $g_\delta \in S(\mathcal{R})$ and a set $G_\delta \in \mathcal{R}$ such that $m_p(G_\delta^c) < 1/n$ and $||g - g_\delta||_{G_\delta} < 1/n$. Then $g_\delta \to f$ in measure. Indeed, let $p_o \in cs(E)$ and $\alpha, \epsilon > 0$. Choose $n_o > 1/\alpha, 1/\epsilon$ and set $\delta_o = (n_o, p_o)$. If $\delta = (n, p) \geq \delta_o$, then

$$\begin{aligned} m_{p_o}^*(\{x : |g_\delta(x) - f(x)| \geq \alpha\} &\leq \bar{m}_p(\{x : |g_\delta(x) - f(x)| \geq \alpha\} \\ &\leq \bar{m}_p(\{x : |g_\delta(x) - f(x)| \geq 1/n\} \\ &\leq m_p(G_\delta^c) < 1/n < \epsilon. \end{aligned}$$

This proves that $g_\delta \to f$ in measure. The last part of the Theorem follows from Theorem 4.4.

Corollary 4.17 *Let $m \in M_\tau(\mathcal{R}, E)$, where E is metrizable and $\mathcal{R}$ a σ-algebra, and let $f \in \mathbb{K}^X$. Then f is measurable iff there exists a sequence (h_n) in $S(\mathcal{R})$ converging in measure to f.*

Proof : The necessity follows from the preceding Theorem. Conversely let (h_n) in $S(\mathcal{R})$ converging in measure to f. By Theorem 4.6, there exist a subsequence (h_{n_k}) and $F \in \mathcal{R}$ such that F is a support set for m and $h_{n_k} \to f$ pointwise on F. Hence f is measurable by Theorem 3.11.

Theorem 4.18 (**Lusin's Theorem**) *Let $m \in M_\tau(\mathcal{R}, E)$, where E is metrizable and $\mathcal{R}$ a σ-algebra, and let $f \in \mathbb{K}^X$. Then f is measurable iff, for each $p \in cs(E)$ and each $\epsilon > 0$, there exist $A \in \mathcal{R}$, with $m_p(A^c) < \epsilon$, and a $\tau_\mathcal{R}$-continuous function g such that $f(x) = g(x)$ for all $x \in A$.*

Proof : Suppose that f is measurable and let $p \in cs(E)$, $\epsilon > 0$. By the preceding Corollary, there exists a sequence (h_n) in $S(\mathcal{R})$ which converges in measure to f. Each h_n is measurable. By theorem 4.6 there exist a subsequence $(g_k) = (h_{n_k})$ and $F \in \mathcal{R}$ such that F is a support set for m and $g_k \to f$ pointwise on F. By Egoroff's Theorem, there exists $A \in \mathcal{R}$, with $m_p(A^c) < \epsilon$, such that $g_k \to f$ uniformly on A. Since A is $\tau_\mathcal{R}$-open and each g_k is $\tau_\mathcal{R}$-continuous, it follows that f is $\tau_\mathcal{R}$-continuous at every point of A. If $g = \chi_A f$, then g is $\tau_\mathcal{R}$-continuous and $g = f$ on A. Conversely, suppose that the condition is satisfied and let B be a clopen subset of $\mathbb{K}$ and $p \in cs(E)$. We need to show that $f^{-1}(B) \in \mathcal{R}_m$. For each positive integer k, there exist $A_k \in \mathcal{R}$, with $m_p(A_k^c) < 1/k$, and a $\tau_\mathcal{R}$-continuous function u_k such that $u_k = f$ on A_k. Let

$$A = \bigcup_k A_k, \quad F = f^{-1}(B) \cap A, \quad G = f^{-1}(B) \cap A^c.$$

Then

$$F = \bigcup_{k=1}^{\infty} f^{-1}(B) \cap A_k = \bigcup_{k=1}^{\infty} u_k^{-1}(B) \cap A_k.$$

Since u_k is $\tau_{\mathcal{R}}$-continuous (and hence $\tau_{\mathcal{R}_m}$-continuous), it follows that u_k is m-measurable and so $F \in \mathcal{R}_m$. Moreover, $G \subset A_k^c$, for each k, and so

$$f^{-1}(B) \triangle F = G \subset A_k^c,$$

which implies that $d_p(f^{-1}(B), F) \leq m_p(A_k^c) < 1/k \to 0$. This proves that $f^{-1}(B)$ belongs to the closure of $\mathcal{R}_m$ in $P(X)$ and hence $f^{-1}(B) \in \mathcal{R}_m$. This completes the proof.

Definition 4.19 *Let $m \in M(\mathcal{R}, E)$. A sequence (f_n) in $\mathbb{K}^X$ is said to be Cauchy in measure if, for each $p \in cs(E)$ and each $\alpha > 0$, we have*

$$\lim_{n,r\to\infty} m_p^*(\{x : |f_n(x) - f_r(x)| \geq \alpha\}) = 0.$$

We have the following easily verified

Lemma 4.20 *If $f_n \to f$ in measure, then (f_n) is Cauchy in measure.*

Theorem 4.21 *Let $m \in M_\sigma(\mathcal{R}, E)$ and suppose that E is metrizablre and $\mathcal{R}$ a σ-algebra. If (f_n) is a sequence of measurable functions which is Cauchy in measure, then there exists an f such that $f_n \to f$ in measure.*

Proof: Let (p_n) be an increasing sequence of continuous seminorms on E such that, for each $p \in cs(E)$, the exists an n with $p \leq p_n$. There are $n_1 < n_2 < \ldots$ such that

$$m_{p_k}^*(\{x : |f_n(x) - f_r(x)| \geq 1/k\}) < 1/k$$

for all $n, r \geq n_k$. Let $h_k = f_{n_k}$ and let $A_k \in \mathcal{R}$ such that $m_{p_k}(A_k) < 1/k$ and

$$\{x : |h_k(x) - h_{k+1}(x)| \geq 1/k\} \subset A_k.$$

Let $F_k = \bigcup_{i \geq k} A_i$. Then $F_k \in \mathcal{R}$ and

$$m_{p_k}(F_k) = \sup_{i \geq k} m_{p_k}(A_i) \leq \sup_{i \geq k} m_{p_i}(A_i) \leq 1/k.$$

On each $X \setminus F_k$, the sequence (h_j) converges uniformly. In fact, let $\epsilon > 0$ and choose $n_o > k, 1/\epsilon$. If $i, j \geq n_o$, then for $x \notin F_k$ we have $|h_i(x) - h_j(x)| < 1/n_o < \epsilon$. It follows now that the $\lim h_j(x)$ exists for every $x \notin F = \bigcap F_k$. Define f on X by $f(x) \doteq \lim h_j(x)$ when $x \notin F$ and arbitrarily when $x \in F$. We will show that $f_n \to f$ in measure. Indeed, let $p \in cs(E)$, $\alpha > 0$ and $\epsilon > 0$. Set

$$B_n = \{x : |f_n(x) - f(x)| \geq \alpha\}.$$

Choose $r > 1/\epsilon$ such that $p \leq p_{n_r}$ and $n_r > 1/\alpha$. Since $h_j \to f$ uniformly on $F_{n_r}^c$, there exists $j \geq r, 1/\alpha$ such that $|h_j(x) - f(x)| < \alpha$ for all $x \in F_{n_r}^c$. Let now $n \geq n_j$. Then $B_n \subset G_1 \bigcup G_2$, where

$$G_1 = \{x : |f_n(x) - f_{n_j}(x)| \geq \alpha\}, \quad \text{and} \quad G_2 = \{x : |f_{n_j}(x) - f(x)| \geq \alpha\}.$$

Moreover $G_2 \subset F_{n_r}$ and so

$$m_p^*(G_2) \leq m_p(F_{n_r}) \leq m_{p_{n_r}}(F_{n_r}) < 1/r < \epsilon.$$

Also

$$G_1 \subset \{x : |f_n(x) - f_{n_j}(x)| \geq 1/j\}$$

and thus

$$m_p^*(G_1) \leq m_{p_{n_j}}(G_1) < 1/j < \epsilon.$$

Hence $m_p^*(B_n) < \epsilon$ for all $n \geq n_j$. This clearly completes the proof.

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Extensions of p-Adic Vector Measures

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Abstract

For $\mathcal{R}$ be a separating algebra of subsets of a set X, E a complete Hausdorff non-Archimedean locally convex space and $m : \mathcal{R} \to E$ a bounded finitely additive measure, it is shown that:
a. If m is σ-additive and strongly additive, then m has a unique σ-additive extension m^σ on the σ-algebra $\mathcal{R}^\sigma$ generated by $\mathcal{R}$.
b. If m is strongly additive and τ-additive, then m has a unique τ-additive extension m^τ on the σ-algebra $\mathcal{R}^{bo}$ of all $\tau_{\mathcal{R}}$-Borel sets, where $\tau_{\mathcal{R}}$ is the topology having $\mathcal{R}$ as a basis.
Also, some other results concerning such measures are given.

1 Preliminaries

Throughout this paper, $\mathbb{K}$ will be a complete non-Archimedean valued field, whose valuation is non-trivial. By a seminorm, on a vector space over $\mathbb{K}$, we will mean a non-Archimedean seminorm. Similarly, by a locally convex space we will mean a non-Archimedean locally convex space over $\mathbb{K}$ (see [12] or [13]). For E a locally convex space, we will denote by cs(E) the collection of all continuous seminorms on E. For X a set, $f \in \mathbb{K}^X$ and $A \subset X$, we define

$$\|f\|_A = \sup\{|f(x)| : x \in A\} \quad \text{and} \quad \|f\| = \|f\|_X.$$

Also for $A \subset X$, A^c will be its complement in X and χ_A the $\mathbb{K}$-valued characteristic function of A. The family of all subsets of X will be denoted by $P(X)$.

Assume next that X is a non-empty set and $\mathcal{R}$ a separating algebra of subsets of X, i.e. $\mathcal{R}$ is a family of subsets of X such that

1. $X \in \mathcal{R}$, and, if $A, B \in \mathcal{R}$, then $A \cup B$, $A \cap B$, A^c are also in $\mathcal{R}$.
2. If x, y are distinct elements of X, then there exists a member of $\mathcal{R}$ which contains x but not y.

Then $\mathcal{R}$ is a base for a Hausdorff zero-dimensional topology $\tau_{\mathcal{R}}$ on X. For E a locally convex space, we denote by $M(\mathcal{R}, E)$ the space of all finitely-additive measures $m : \mathcal{R} \to E$ such that $m(\mathcal{R})$ is a bounded subset of E (see [10]). For a net (V_δ) of subsets of X, we write $V_\delta \downarrow \emptyset$ if (V_δ) is decreasing and $\cap V_\delta = \emptyset$. An element $m \in M(\mathcal{R}, E)$ is said to be σ-additive if $m(V_n) \to 0$ for each sequence (V_n) in $\mathcal{R}$ which decreases to the empty set. We denote by $M_\sigma(\mathcal{R}, E)$ the space of all σ-additive members of $M(\mathcal{R}, E)$. An m of $M(\mathcal{R}, E)$ is said to be τ-additive if $m(V_\delta) \to 0$ for each net (V_δ) in $\mathcal{R}$ with $V_\delta \downarrow \emptyset$. We will denote by $M_\tau(\mathcal{R}, E)$ the space of all τ-additive members of $M(\mathcal{R}, E)$. For $m \in M(\mathcal{R}, E)$ and $p \in cs(E)$, we define

$$m_p : \mathcal{R} \to \mathbb{R}, \quad m_p(A) = \sup\{p(m(V)) : V \in \mathcal{R}, V \subset A\} \quad \text{and} \quad \|m\|_p = m_p(X).$$

We also define

$$N_{m,p} : X \to \mathbb{R}, \quad N_{m,p}(x) = \inf\{m_p(V) : x \in V \in \mathcal{R}\}.$$

Next we will recall the definition of the integral of an $f \in \mathbb{K}^X$ with respect to some $m \in M(\mathcal{R}, E)$. Assume that E is a complete Hausdorff locally convex space. For $A \subset X$, let $\mathcal{D}_A$ be the family of all $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\}$, where $\{A_1, A_2, \ldots, A_n\}$ is an $\mathcal{R}$-partition of A and $x_k \in A_k$. We make $\mathcal{D}_A$ into a directed set by defining $\alpha_1 \geq \alpha_2$ if the partition of A in α_1 is a refinement of the one in α_2. For $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\}$, we define $\omega_\alpha(f, m) = \sum_{k=1}^n f(x_k) m(A_k)$. If the limit $\lim \omega_\alpha(f, m)$ exists in E, we will say that f is m-integrable over A and denote this limit by $\int_A f\, dm$. For $A = X$, we write simply $\int f\, dm$. It is easy to see that if f is m-integrable over X, then it is m-integrable over every $A \in \mathcal{R}$ and $\int_A f\, dm = \int \chi_A f\, dm$. If f is bounded on A, then

$$p\left(\int_A f\, dm\right) \leq \|f\|_A \cdot m_p(A).$$

2 Strongly Additive Measures

Throughout the paper, $\mathcal{R}$ will be a separating algebra of subsets of a set X, E a complete Hausdorff locally convex space and $M(\mathcal{R}, E)$ the space of all bounded E-valued finitely-additive measures on $\mathcal{R}$. We will denote by $\tau_{\mathcal{R}}$ the topology on X which has $\mathcal{R}$ as a basis. Every member of $\mathcal{R}$ is $\tau_{\mathcal{R}}$-clopen, i.e both closed and open. By $S(\mathcal{R})$ we will denote the space of all $\mathbb{K}$-valued $\mathcal{R}$-simple functions. As in [10], if $m \in M(\mathcal{R}, E)$, then a subset A of X is said

to be m-measurable if the characteristic function χ_A is m-integrable. By [10, Theorem 4.7], A is measurable iff, for each $p \in cs(E)$ and each $\epsilon > 0$, there exist V, W in $\mathcal{R}$ such that $V \subset A \subset W$ and $m_p(W \setminus V) < \epsilon$.
Let $\mathcal{R}_m$ be the family of all m-measurable sets. The following Theorem gives some results contained in [10] which are needed for the paper.

Theorem 2.1 *1. $\mathcal{R}_m$ is an algebra of subsets of X.*

2. If $\bar{m} : \mathcal{R}_m \to E$, $\bar{m}(A) = \int \chi_A \, dm$, then $\bar{m} \in M(\mathcal{R}_m, E)$.

3. $\bar{m}$ is σ-additive iff m is σ-additive.

4. $\bar{m}$ is τ-additive iff m is τ-additive.

5. For $p \in cs(E)$, we have $N_{m,p} = N_{\bar{m},p}$.

6. $\mathcal{R}_m = \mathcal{R}_{\bar{m}}$.

7. For $A \in \mathcal{R}$, we have $m_p(A) = \bar{m}_p(A)$.

8. For $A \in \mathcal{R}_m$, we have

$$\bar{m}_p(A) = \inf\{m_p(W) : W \in \mathcal{R}, A \subset W\}.$$

9. If m is σ-additive and $V_n \downarrow \emptyset$, then $m_p(V_n) \to 0$.

10. If m is τ-additive and $V_\delta \downarrow \emptyset$, then $m_p(V_\delta) \to 0$.

11. If m is σ-additive and if (V_n) is a sequence in $\mathcal{R}$, then for every set V in $\mathcal{R}$ contained in $\bigcup V_n$, we have $m_p(V) \leq \sup_n m_p(V_n)$.

12. If m is τ-additive and if (V_δ) is a family in $\mathcal{R}$, then for every set V in $\mathcal{R}$ contained in $\bigcup V_\delta$, we have $m_p(V) \leq \sup_\delta m_p(V_\delta)$.

13. An $f \in \mathbb{K}^X$ is m-integrable iff, for each $p \in cs(E)$ and each $\epsilon > 0$, there exists an $\mathcal{R}$-partition $\{A_1, \ldots, A_n\}$ of X such that, for each $1 \leq k \leq n$, we have $|f(x) - f(y)| \cdot m_p(A_k) \leq \epsilon$ if $x, y \in A_k$. In this case, if $x_k \in A_k$, then

$$p\left(\int f \, dm - \sum_{k=1}^{n} f(x_k) m(A_k)\right) \leq \epsilon.$$

14. If m is τ-additive, then a subset A of X is measurable iff A is $\tau_{\mathcal{R}_m}$-clopen.

Definition 2.2 *An element m of $M(\mathcal{R}, E))$ is said to be strongly additive if, for each sequence (A_n) of pairwise disjoint members of $\mathcal{R}$, we have that $m(A_n) \to 0$.*

It is clear that, if $\mathcal{R}$ is a σ-algebra and m σ-additive, then m is strongly additive.

Theorem 2.3 *For an $m \in M(\mathcal{R}, E)$, the following are equivalent :*

1. *m is strongly additive.*
2. *For each decreasing sequence (A_n) of members of $\mathcal{R}$, the sequence $(m(A_n))$ is convergent in E.*
3. *For each sequence (A_n) of pairwise disjoint members of $\mathcal{R}$ and each $p \in cs(E)$, we have $m_p(A_n) \to 0$.*
4. *For each decreasing net (V_δ) in $\mathcal{R}$, the net $(m(V_\delta))$ converges in E.*
5. *For each decreasing net (V_δ) in $\mathcal{R}$, each $p \in cs(E)$ and each $\epsilon > 0$, there exists δ_o such that $m_p(V_\delta \triangle V_{\delta'}) < \epsilon$ for all δ, $\delta' \geq \delta_o$.*
6. *For each family $(V_i)_{i \in I}$, of pairwise disjoint members of $\mathcal{R}$, each $p \in cs(E)$ and each $\epsilon > 0$, there exists $J \subset I$ finite such that $m_p(V_i) < \epsilon$ for all $i \notin J$.*
7. *Let $(V_i)_{i \in I}$ be a family of pairwise disjoint members of $\mathcal{R}$. For $J \subset I$ finite, let $W_J = \bigcup_{i \in J} V_i$. Then the net $(m(W_J))$ is convergent.*

Proof : (1) $\Rightarrow$ (3). Assume the contrary. Then, there exist $p \in cs(E)$, $\epsilon > 0$ and $n_1 < n_2 < \ldots$ such that $m_p(A_{n_k}) > \epsilon$ for all k. For each k, there exisis a B_k contained in A_{n_k} such that $p(m(B_k)) > \epsilon$. This contradicts our hypothesis (1).
(3) $\Rightarrow$ (5). Assume that (5) does not hold. Then, there exist $p \in cs(E)$ and $\epsilon > 0$ such that, for each δ there are δ_1, $\delta_2 \geq \delta$ with $m_p(V_{\delta_1} \triangle V_{\delta_2}) > \epsilon$. Thus, for each δ, there exists $\delta' \geq \delta$ such that $m_p(V_\delta \setminus V_{\delta'}) > \epsilon$. Now, there exist $\delta_1 \leq \delta_2 \leq \ldots$ such that $m_p(V_{\delta_k} \setminus V_{\delta_{k+1}}) > \epsilon$ for all k. If $G_n = V_{\delta_n} \setminus V_{\delta_{n+1}}$, then the sets G_n are pairwise disjoint, which contradicts (3).
(5) $\Rightarrow$ (4). Let (V_δ) be a decreasing net in $\mathcal{R}$ and $p \in cs(E)$. Then, for all δ, δ', we have $p(m(V_\delta) - m(V_{\delta'}) \leq m_p(V_\delta \triangle V_{\delta'})$. This, by our hypothesis, implies that the net $(m(V_\delta))$ is Cauchy and hence convergent.
(4) $\Rightarrow$ (2). It is trivial.
(2) $\Rightarrow$ (1). For (A_n) a sequence of pairwise disjoint members of $\mathcal{R}$, let

$$B_n = \left(\bigcup_{k=-1}^{n} A_k \right)^c .$$

Then (B_n) is decreasing and so the sequence $(m(B_n))$ is convergent. Thus, given p
in $cs(E)$ and $\epsilon > 0$, there exists n_o such that

$$p(m(B_n \setminus B_{n+1})) = p(m(B_n) - m(B_{n+1})) < \epsilon$$

for $n \geq n_o$. But $B_n \setminus B_{n+1} = A_{n+1}$. Thus $m(A_n) \to 0$
(3) $\Rightarrow$ (6). Let $(V_i)_{i \in I}$ be a family of pairwise disjoint members of $\mathcal{R}$ and suppose that, for some $p \in cs(E)$ and some $\epsilon > 0$, the set $\{I \in I : m_p(V_i) \geq \epsilon\}$ is infinite. Hence there are distinct i_k, $k = 1, 2, \ldots$, such that $m_p(V_{i_k}) \geq \epsilon\}$, which contradicts our hypothesis (3).
(6) $\Rightarrow$ (7). Let $(V_i)_{i \in I}$ be a family of pairwise disjoint members of $\mathcal{R}$. For $J \subset I$ finite, let $W_J = \bigcup_{i \in J} V_i$. Let J_o be a finite subset of I such that $m_p(V_i) < \epsilon$. If now J is any finite subset of I containing J_o, then

$$p(m(W_J) - m(W_{J_o})) = p(\bigcup_{i \in J \setminus J_o} V_i) \leq \max_{i \in J \setminus J_o} m_p(V_i) < \epsilon.$$

Hence the net $(m(W_J))$ is Cauchy and therefore convergent.
(7) $\Rightarrow$ (1). It follows easily.

Definition 2.4 *A family H of members of $M(\mathcal{R}, E)$, is said to be uniformly strongly additive iff, for each sequence (A_n), of pairwise disjoint members of $\mathcal{R}$, we have that $m(A_n) \to 0$ uniformly for $m \in H$.*

Using arguments analogous to the ones used in the proof of Theorem 2.3, we get the following

Theorem 2.5 *For a subset H of $M(\mathcal{R}, E)$, the following are equivalent:*

1. *H is uniformly strongly additive.*
2. *For each sequence (A_n), of pairwise disjoint members of $\mathcal{R}$, and each $p \in cs(E)$, we have that*

$$\lim_{n \to \infty} m_p(A_n) = 0$$

   uniformly for $m \in H$.

3. *If (A_n) is a decreasing sequence of members of $\mathcal{R}$, then, for each $p \in cs(E)$ and each $\epsilon > 0$, there exists n_o such that $m_p(A_n \setminus A_k) \leq \epsilon$ for all $k > n \geq n_o$.*

Theorem 2.6 *Let $H \subset M_\sigma(\mathcal{R}, E)$ be uniformly strongly additive and let (A_n) be a sequence in $\mathcal{R}$ such that $A_n \downarrow \emptyset$. Then, for each $p \in cs(E)$, $m_p(A_n) \to 0$ uniformly for $m \in H$.*

Proof : Given $p \in cs(E)$ and $\epsilon > 0$, there exists n_o such that $m_p(A_n \setminus A_k) < \epsilon$ for all $k > n \geq n_o$ and each $m \in H$. Let now $n \geq n_o$. For $k > n$, we have $A_n = (A_n \setminus A_k) \bigcup A_k$ and so

$$m_p(A_n) = \max\{m_p(A_n \setminus A_k), m_p(A_k)\} \leq \max\{\epsilon, m_p(A_k)\}.$$

Since $m_p(A_k) \to 0$ when $k \to \infty$, it follows that $m_p(A_n) \leq \epsilon$ for all $n \geq n_o$ and all $m \in H$. This completes the proof.

Theorem 2.7 (**Nikodym Boundedness Theorem**) *Assume that $\mathcal{R}$ is a σ-algebra and let H be a subset of $M(\mathcal{R}, E)$ consisting of strongly additive measures. If, for each $A \in \mathcal{R}$, the set $H(A) = \{m(A) : m \in H\}$ is bounded in E, then the set $H(\mathcal{R}) = \{m(A) : A \in \mathcal{R}, m \in H\}$ is bounded, equivalently $\sup_{m \in H} \|m\|_p < \infty$ for each $p \in cs(E)$.*

Proof : Assume the contrary. Then, there exist a $p \in cs(E)$ and a sequence (m_n) in H such that $\sup_n \|m_n\|_p = \infty$.
Claim I: If $G \in \mathcal{R}$ is such that $\sup_n (m_n)_p(G) = \infty$, then, for each $\alpha > 0$, there exist an n and an $\mathcal{R}$-partition $\{A, B\}$ of G such that $p(m_n(A)) = p(m_n(B)) > \alpha$. Indeed, there exist an n and $A \in \mathcal{R}$, $A \subset G$, such that

$$p(m_n(A)) > \max\{\alpha, \sup_k p(m_k(G))\} \geq \max\{\alpha, p(m_n(G))\}.$$

If $B = G \setminus A$, then

$$p(m_n(A)) > p(m_n(G)) = p(m_n(A) + m_n(B)).$$

Thus $p(m_n(A)) = p(m_n(B)) > \alpha$. Let now n_1 be a positive integer and $\{A_1, B_1\}$ an $\mathcal{R}$-partition of X such that $p(m_{n_1}(A_1)) = p(m_{n_1}(B_1)) > 1$. One of the $\sup_n (m_n)_p(A_1)$, $\sup_n (m_n)_p(B_1)$ must be infinite. If the former is infinite, take $G_1 = A_1$ and $F_1 = B_1$, otherwise take $G_1 = B_1$ and $F_1 = A_1$. Let $n_2 > n_1$ and $\{A_2, B_2\}$ an $\mathcal{R}$-partition of G_1 such that

$$p(m_{n_2}(A_2)) = p(m_{n_2}(B_2)) > \max\{2, \sup_k p(m_k(F_1))\} \geq \max\{2, p(m_{n_2}(F_1))\}.$$

One of the $\sup_n (m_n)_p(A_2)$, $\sup_n (m_n)_p(B_2)$ must be infinite. If the former is infinite, take $G_2 = A_2$ and $F_2 = B_2$, otherwise take $G_2 = B_2$ and $F_2 = A_2$. We continue using the same argument and get by induction a sequence (F_k), of pairwise disjoint members of $\mathcal{R}$, and $n_1 < n_2 < \ldots$ such that

$$p(m_{n_k}(F_k)) > \max\{k, \max_{1 \leq j < k} p(m_{n_k}(F_j))\}.$$

Let $\mu_k = m_{n_k}$. **Claim II** : For each $m \in H$ and each infinite subset Ω of $\mathbb{N}$, there exists an infinite subset Z of Ω such that $m_p\left(\bigcup_{n \in Z} F_n\right) < 1$. Indeed, there exists an infinite partition $\Omega_1, \Omega_2, \ldots$ of Ω into infinite sets. The sets $D_k = \bigcup_{n \in \Omega_k} F_n$, $k \in \mathbb{N}$, are pairwise disjoint members of $\mathcal{R}$. Since m is strongly additive, there exists a k such that $m_p(D_k) < 1$.
Let now $r_1 = 1$, $W_{1,j} = F_j$ for $j \in \mathbb{N}$. By the preceding Claim, there exists a subsequence $(W_{2,j})$ of $(W_{1,j})$, with $W_{2,1} = F_{r_2}$ and $r_2 > r_1$, such that

$$(\mu_{r_1})_p \left(\bigcup_j W_{2,j} \right) < 1.$$

Next, there exists a subsequence $(W_{3,j})$ of $(W_{2,j})$, with $W_{3,1} = F_{r_3}$, and $r_3 > r_2$, such that

$$(\mu_{r_2})_p \left(\bigcup_j W_{3,j} \right) < 1.$$

We continue the same argument using induction. Let $W = \bigcup_{i=1}^{\infty} F_{r_i}$. For each j we have

$$\mu_{r_j}(W) = \mu_{r_j}(F_{r_j}) + \mu_{r_j}\left(\bigcup_{k<j} F_{r_k} \right) + \mu_{r_j}\left(\bigcup_{k>j} F_{r_k} \right).$$

Now $p\left[\mu_{r_j}(F_{r_j})\right] > r_j \geq 1$, $(\mu_{r_j})_p \left(\bigcup_{k>j} F_{r_k} \right) < 1$ and

$$p\left(\mu_{r_j}(\bigcup_{k<j} F_{r_k}) \right) < p(\mu_{r_j}(F_{r_j})).$$

Thus $p(\mu_{r_j}(W)) = p(\mu_{r_j}(F_{r_j})) > r_j$ and so $\sup_j p(\mu_{r_j}(W)) = \infty$, a contradiction. This completes the proof.

Theorem 2.8 *If $m \in M(\mathcal{R}, E)$ is strongly additive, then $\bar{m} \in M(\mathcal{R}_m, E)$ is also strongly additive.*

Proof : Let (A_n) be a sequence, of pairwise disjoint members of $\mathcal{R}_m$, and let $p \in cs(E)$ and $\epsilon > 0$. For each n, there exist V_n, W_n in $\mathcal{R}$ such that $V_n \subset A_n \subset W_n$ and $m_p(W_n \setminus V_n) < \epsilon$. As m is strongly additive, there exists n_o such that $m_p(V_n) < \epsilon$ for each $n \geq n_o$. If now $n \geq n_o$, then $A_n \subset V_n \bigcup (W_n \setminus V_n)$ and so

$$\bar{m}_p(A_n) \leq \max\{m_p(W_n \setminus V_n), m_p(V_n)\} < \epsilon.$$

Hence $\bar{m}$ is strongly additive.

3 Absolute Continuity

Definition 3.1 *An element m of $M(\mathcal{R}, E)$ is said to be absolutely continuous with respect to some $\mu \in M(\mathcal{R})$, and write $m \ll \mu$, if*

$$\lim_{|\mu|(A) \to 0} m_p(A) = 0$$

for each $p \in cs(E)$. Equivalently, for each $p \in cs(E)$ and each $\epsilon > 0$, there exists $\delta > 0$ such that $m_p(A) < \epsilon$ for each $A \in \mathcal{R}$ with $|\mu|(A) < \delta$.

Theorem 3.2 *Let $\mu \in M_\sigma(\mathcal{R})$ and $m \in M_\sigma(\mathcal{R}, E)$. If $\mathcal{R}$ is a σ-algebra, then $m \ll \mu$ iff $m(A) = 0$ for each $A \in \mathcal{R}$ with $|\mu|(A) = 0$.*

Proof : The condition is clearly necessary. Conversely, suppose that the condition is satisfied but m is not μ-absolutely continuous. Then there exist $p \in cs(E)$ and $\epsilon > 0$ and a sequence (A_n) in $\mathcal{R}$, with $|\mu|(A_n) < 1/n$, such that $m_p(A_n) > \epsilon$ for all n. let $G_n = \bigcup_{k \geq n} A_k$, $G = \bigcap G_n$. Then

$$|\mu|(G) \leq |\mu|(G_n) = \sup_{k \geq n} |\mu|(A_k) < 1/n \to 0.$$

By our hypothesis, $m_p(G) = 0$. Let $G_o = X$ and $B_n = G_{n-1} \setminus G_n$ for each $n \in \mathbb{N}$. The sequence (B_n) consists of pairwise disjoint members of $\mathcal{R}$. Moreover $G_n \setminus G = \bigcup_{k>n} B_k$ and so $m_p(G_n \setminus G) \to 0$. Also, $A_n \subset G_n = G \bigcup (G_n \setminus G)$ and hence

$$m_p(A_n) \leq \max\{m_p(G), m_p(G_n \setminus G)\} = m_p(G_n \setminus G) \to 0$$

as $n \to \infty$. This completes the proof

Theorem 3.3 *Let $m \in M(\mathcal{R}, E)$ and $\mu \in M(\mathcal{R})$ be such that $m \ll \mu$. Then :*

1. *$\mathcal{R}_\mu \subset \mathcal{R}_m$.*
2. *If $m_1 = \bar{m}|_{\mathcal{R}_\mu}$, then $m_1 \ll \bar{\mu}$.*

Proof : 1. Assume that $A \in \mathcal{R}_\mu$ and let $p \in cs(E)$ and $\epsilon > 0$. Since $m \ll \mu$, there exists $\delta > 0$ such that $m_p(B) < \epsilon$ if $|\mu|(B) < \delta$. As $A \in \mathcal{R}_\mu$, there are $V, W \in \mathcal{R}$, with $V \subset A \subset W$, such that $|\mu|(W \setminus V) < \delta$. But then $m_p(W \setminus V) < \epsilon$, which proves that A is in $\mathcal{R}_m$.
2. Let $p \in cs(E)$ and $\epsilon > 0$. There exists $\delta > 0$ such that $m_p(B) < \epsilon$ if $|\mu|(B) < \delta$. Let now $A \in \mathcal{R}_\mu$ with $(\bar{\mu})_p(A) < \delta$. We will show that $\bar{m}_p(A) < \epsilon$. In fact, there are $V, W \in \mathcal{R}$, with $V \subset A \subset W$, such that $|\mu|(W \setminus V) < \delta$. But then,

$$\bar{m}_p(W \setminus V) = m_p(W \setminus V) < \epsilon.$$

Also

$$|\mu|(V) = |\bar{\mu}|(V) \leq |\bar{\mu}|(A) < \delta$$

and hence $\bar{m}_p(V) = m_p(V) < \epsilon$. Since $A \subset V \bigcup (W \setminus V)$, we have that

$$\bar{m}_p(A) \leq \max\{\bar{m}_p(V), \bar{m}_p(W \setminus V)\} < \epsilon.$$

Hence the result follows.

Definition 3.4 *Let $\mu \in M(\mathcal{R})$. A collection $H \subset M(\mathcal{R}, E)$ is said to be uniformly absolutely continuous with respect to μ if, for each $p \in cs(E)$, we have*

$$\lim_{|\mu|(A) \to 0} \sup_{m \in H} m_p(A) = 0.$$

Theorem 3.5 *Let H be a uniformly strongly additive subset of $M(\mathcal{R}, E)$ and let $\mu \in M(\mathcal{R})$ be such that $m \ll \mu$ for each $m \in H$. Then H is uniformly absolutely continuous with respect to μ.*

Proof : Assume the contrary. Then, there exist $p \in cs(E)$ and $\epsilon > 0$ such that, for each $\delta > 0$, there are $m \in H$ and $A \in \mathcal{R}$ such that $|\mu|(A) < \delta$ and $m_p(A) > \epsilon$. Let $A_1 \in \mathcal{R}$, $m_1 \in H$, $\delta_1 = 1$ be such that $|\mu|(A_1) < \delta_1$ and $(m_1)_p(A_1) > \epsilon$. Since $m_1 \ll \mu$, there exists $\delta_2 > 0$ such that, if $|\mu|(A) < \delta_2$, then $(m_1)_p(A) < \epsilon$. There exist $A_2 \in \mathcal{R}$ and $m_2 \in H$ such that $|\mu|(A_2) < \delta_2$ and $(m_2)_p(A_2) > \epsilon$. Next there exists $\delta_3 > 0$ such that, if $|\mu|(A) < \delta_3$, then $(m_1)_p(A) < \epsilon$ and $(m_2)_p(A) < \epsilon$. Let $m_3 \in H$ and $A_3 \in \mathcal{R}$ be such that $|\mu|(A_3) < \delta_3$ and $(m_3)_p(A_3) > \epsilon$. Inductively, we get a sequence (m_n) in H and a sequence (A_n) in $\mathcal{R}$ such that $(m_n)_p(A_n) > \epsilon$ and $(m_k)_p(A_n) < \epsilon$ if $k < n$.
Claim There are $n_o = 1 < n_1 < n_2 < \ldots < n_k$ such that, for $G_o = A_1$, $G_1 = G_o \setminus A_1, \ldots, G_k = G_{k-1} \setminus A_k$, we have

1. $(m_{n_j})_p(G_{j-1} \cap A_{n_j}) > \epsilon$, for $j = 1, 2, \ldots, k$.
2. $(m_n)_p(G_k \cap A_n) \le \epsilon$ for every $n > n_k$.

In fact, if $(m_n)_p(G_o \cap A_n) \le \epsilon$ for every $n > 1$, take $k = 0$, $n_o = 1$. Otherwise, choose $n_1 > n_o = 1$ such that $(m_{n_1})_p(G_o \cap A_{n_1}) > \epsilon$ and let $G_1 = G_o \setminus A_{n_1}$. If $(m_n)_p(G_1 \cap A_n) \le \epsilon$ for all $n > n_1$, take $k = 1$. Otherwise, choose $n_2 > n_1$ such that $(m_{n_2})_p(G_1 \cap A_{n_2}) > \epsilon$ and let $G_2 = G_1 \setminus A_{n_2}$. If this process does not eventually terminate, we find by induction $n_o = 1 < n_1 < n_2 \ldots$ such that, for $G_o = A_1$ and $G_k = G_{k-1} \setminus A_{n_k}$, for $k \ge 1$, we have $(m_{n_k})_p(G_{k-1} \cap A_{n_k}) > \epsilon$ for all $k \ge 1$. Let $D_k = G_{k-1} \setminus G_k$, $k \ge 1$. The sets D_k are pairwise disjoint. Moreover, $D_k = G_{k-1} \cap A_{n_k}$ and so $(m_{n_k})_p(D_k) > \epsilon$, for all k, which contradicts the fact that H is uniformly strongly additive. Hence the claim holds. Let now $n_o = 1 < n_1 < n_2 < \ldots < n_k$ be as in the claim. Since

$$A_1 = \left[\bigcup_{j=1}^{k} A_1 \cap A_{n_j} \right] \bigcup G_k$$

and $(m_1)_p(A_1) > \epsilon$ while $(m_1)_p(A_1 \cap A_{n_j}) \le (m_1)_p(A_{n_j}) < \epsilon$ for $j \le k$, it follows that $(m_1)_p(G_k) > \epsilon$. Let $F_1 = G_k \subset A_1$ and $r_1 = n_k$. For $n > r_1$, let $B_n = A_n \setminus F_1$. Let $n > r_1$. Then $A_n = (A_n \cap F_1) \cup B_n$. Since $(m_n)_p(A_n \cap F_1) \le \epsilon$ and $(m_n)_p(A_n) > \epsilon$, it follows that $(m_n)_p(B_n) > \epsilon$. Also, for $r_1 < n < N$, we have $(m_n)_p(B_N) \le (m_n)_p(A_N) < \epsilon$. Now, we can apply the same argument as above, replacing (A_n) by $(B_n)_{n>r_1}$ and (m_n) by $(m_n)_{n>r_1}$. We will then get an $r_2 > r_1$ and $F_2 \subset B_1$ such that $(m_{r_1+1})_p(F_2) > \epsilon$ and $(m_n)_p(F_2 \cap B_n) \le \epsilon$ for all $n > r_2$. For $n > r_2$, let $Z_n = B_n \setminus F_2$. Since $B_n = (B_n \cap F_2) \cup Z_n$ and since $(m_n)_p(B_n \cap F_2) \le \epsilon$, while $(m_n)_p(B_n) > \epsilon$, we get that $(m_n)_p(Z_n) > \epsilon$. Also, for $r_2 < n < N$, we have $(m_n)_p(Z_N) \le (m_n)_p(B_N) < \epsilon$. Thus we may repeat the same argumrent for the sequences $(Z_n)_{n>r_2}$ and $(m_n)_{n>r_2}$. Inductively, we get a sequence (F_k), of pairwise disjoint members of $\mathcal{R}$, and a sequence (m_k') in H such that $(m_k')_p(F_k) > \epsilon$. This contradicts our hypothesis that H is uniformly strongly additive. Hence the result follows.

4 Extensions of σ-Additive Measures

In this section we will examine the problem of extending an $m \in M(\mathcal{R}, E)$ to a σ-additive measure defined on a σ-algebra containing $\mathcal{R}$. In order for such an extension to exist, it is clearly necessary that m is σ-additive and strongly additive. We will show that these two conditions are also sufficient.
Throughout the rest of this section, m will be a strongly additive member of $M_\sigma(\mathcal{R}, E)$.
For $p \in cs(E)$, we define

$$\hat{m}_p : P(X) \to \mathbb{R}, \quad \hat{m}_p(A) = \inf \sup_n m_p(V_n),$$

where the infimum is taken over the collection of all sequences (V_n) in $\mathcal{R}$ which cover A. It is easy to show that $\hat{m}_p(A \cup B) = \max\{\hat{m}_p(A), \hat{m}_p(B)\}$.

Lemma 4.1 $m_p(A) = \hat{m}_p(A)$ *for all* $A \in \mathcal{R}$.

Proof : Clearly $\hat{m}_p(A) \le m_p(A)$. On the other hand, if (V_n) is a sequence in $\mathcal{R}$ covering A, then $m_p(A) \le \sup_n m_p(V_n)$ since m is σ-additive. This implies that $\hat{m}_p(A) \ge m_p(A)$.

Let now

$$\hat{d}_p : P(X) \times P(X) \to \mathbb{R}, \quad \hat{d}_p(A, B) = \hat{m}_p(A \triangle B),$$

where $A \triangle B = (A \setminus B) \bigcup (B \setminus A)$. Then $\hat{d}_p$ is an ultrapseudometric on $P(X)$. Let $\mathcal{U}_m^\sigma$ be the uniformity induced by the pseudometrics $\hat{d}_p$, $p \in cs(E)$. For the map

$$m : \mathcal{R} \to E,$$

we have $p(m(A) - m(B)) \le \hat{d}_p(A, B)$. Thus m is $\mathcal{U}_m^\sigma$-uniformly continuous and hence there exists a unique uniformly continuous extension

$$m'^{\sigma} : \hat{\mathcal{R}}_m \to E,$$

where $\hat{\mathcal{R}}_m$ is the closure of $\mathcal{R}$ in $P(X)$ with respect to the topology induced by $\mathcal{U}_m^\sigma$.

Lemma 4.2 $\hat{\mathcal{R}}_m$ *is a separating algebra of subsets of* X *and* $m'^{\sigma} \in M(\hat{\mathcal{R}}_m, E)$.

Proof : Let $A, B \in \hat{\mathcal{R}}_m$, $p \in cs(E)$ and $\epsilon > 0$. There are V_1, V_2 in $\mathcal{R}$ such that $\hat{m}_p(A \triangle V_1) < \epsilon$, $\hat{m}_p(B \triangle V_2) < \epsilon$. If $V = V_1 \cup V_2$ and $W = V_1 \cap V_2$, then

$$(A \cup B) \triangle V \subset (A \triangle V_1) \cup (B \triangle V_2), \quad (A \cap B) \triangle W \subset (A \triangle V_1) \cup (B \triangle V_2)$$

and $A^c \triangle V_1^c = A \triangle V_1$. Hence

$$\hat{m}_p((A \cup B) \triangle V) < \epsilon, \quad \hat{m}_p((A \cap B) \triangle W) < \epsilon, \quad \hat{m}_p(A^c \triangle V_1^c) < \epsilon,$$

which proves that the sets $A \cup B$, $A \cap B$ and A^c are in $\hat{\mathcal{R}}_m$. Also $A \setminus B = A \cap B^c \in \hat{\mathcal{R}}_m$ and so $\hat{\mathcal{R}}_m$ is an algebra. Since

$$m'^{\sigma}(\hat{\mathcal{R}}_m) \subset \overline{m(\mathcal{R})}.$$

it follows that m'^{σ} is bounded. Finally we need to show that m'^{σ} is finitely additive. To this end, we consider the set

$$\Delta = \{(p, n) : p \in cs(E), n \in \mathbb{N}\}$$

and make Δ into a directed set by defining $(p_1, n_1) \geq (p_2, n_2)$ iff $p_1 \geq p_2$ and $n_1 \geq n_2$. Let now A, B be disjoint members of $\hat{\mathcal{R}}_m$. For each $\delta = (p, n)$ in Δ, there are V_δ, W_δ in $\mathcal{R}$ such that

$$\hat{m}_p(A \triangle V_\delta) < 1/n, \quad \hat{m}_p(B \triangle W_\delta) < 1/n.$$

Now the nets (V_δ), (W_δ) converge to A, B, respectively, with respect to the uniformity $\mathcal{U}_m^\sigma$. If $Z_\delta = W_\delta \setminus V_\delta$, then $B \triangle Z_\delta \subset (A \triangle V_\delta) \cup (B \triangle W_\delta)$, which implies that $Z_\delta \to B$. If $F_\delta = V_\delta \cup Z_\delta$ and $D = A \cup B$, then

$$D \triangle F_\delta \subset (A \triangle V_\delta) \cup (B \triangle Z_\delta)$$

and hence $F_\delta \to D$. Thus

$$\begin{aligned} m'^{\sigma}(D) &= \lim m(V_\delta \cup Z_\delta) = \lim[m(V_\delta) + m(Z_\delta)] \\ &= \lim m(V_\delta) + \lim m(Z_\delta) = m'^{\sigma}(A) + m'^{\sigma}(B). \end{aligned}$$

This completes the proof.

Lemma 4.3 *1. For $A, B \subset X$, we have*

$$|\hat{m}_p(A) - \hat{m}_p(B)| \leq \hat{m}_p(A \triangle B).$$

2. If $A, B \in \hat{\mathcal{R}}_m$, then

$$|m_p'^{\sigma}(A) - m_p'^{\sigma}(B)| \leq m_p'^{\sigma}(A \triangle B).$$

Proof : 1. Suppose (say) that

$$\hat{m}_p(A) - \hat{m}_p(B)| > \hat{m}_p(A \triangle B).$$

Since $A = (A \cap B) \cup (A \setminus B)$, we have

$$\hat{m}_p(A) = \max\{\hat{m}_p(A \cap B), \hat{m}_p(A \setminus B)\} = \hat{m}_p(A \cap B) \le \hat{m}_p(B),$$

a contradiction.
2. The proof is analogous to that of (1).

Lemma 4.4 *1. For $G \in \mathcal{R}$, we have $m_p'^{\sigma}(G) = m_p(G)$.*
2. If $A \in \hat{\mathcal{R}}_m$, then $m_p'^{\sigma}(A) = \hat{m}_p(A)$.

Proof : 1. It is clear that $m_p(G) \le m_p'^{\sigma}(G)$. On the other hand, let $B \in \hat{\mathcal{R}}_m$ be contained in G. There exists a net (V_δ) in $\mathcal{R}$ converging to B in the uniformity $\mathcal{U}_m^\sigma$. Then $V_\delta \cap G \to B \cap G = B$. Thus

$$p(m'^{\sigma}(B)) = \lim p(m'^{\sigma}(V_\delta \cap G)) = \lim p(m(V_\delta \cap G)) \le m_p(G),$$

which proves that $m_p'^{\sigma}(G) \le m_p(G)$.
2. Let $B \in \hat{\mathcal{R}}_m$, $B \subset A$. There exists a net (W_δ) in $\mathcal{R}$ converging to B. But then $W_\delta \cap A \to B \cap A = B$. Thus

$$p(m'^{\sigma}(B)) = \lim p(m'^{\sigma}(W_\delta \cap A)).$$

Since

$$p(m'^{\sigma}(W_\delta \cap A) \le m_p'^{\sigma}(W_\delta) = \hat{m}_p(W_\delta) \quad \textit{and} \quad \hat{m}_p(W_\delta) \to \hat{m}_p(B) \le \hat{m}_p(A)$$

(by the preceding Lemma), we get that $m_p'^{\sigma}(A) \le \hat{m}_p(A)$. If (V_δ) is a net in $\mathcal{R}$ which converges to A in the uniformity $\mathcal{U}_m^\sigma$, then

$$\begin{aligned} |\hat{m}_p(V_\delta) - m_p'^{\sigma}(A)| &= |m_p'^{\sigma}(V_\delta) - m_p'^{\sigma}(A)| \\ &\le m_p'^{\sigma}(A \triangle V_\delta) \le \hat{m}_p(A \triangle V_\delta) \to 0 \end{aligned}$$

and so $\hat{m}_p(V_\delta) \to m_p'^{\sigma}(A)$. Also $\hat{m}_p(V_\delta) \to \hat{m}_p(A)$, by the preceding Lemma, and hence $m_p'^{\sigma}(A) = \hat{m}_p(A)$.

Lemma 4.5 $N_{m,p} = N_{m'^{\sigma},p}$.

Proof : Suppose that, for some $x \in X$, we have $N_{m,p}(x) > \alpha > N_{m'^{\sigma},p}(x)$. There exists $A \in \hat{\mathcal{R}}_m$, containing x, such that $m_p'^{\sigma}(A) < \alpha$. Let $V \in \mathcal{R}$ be such that $m_p'^{\sigma}(A \triangle V) = \hat{m}_p(A \triangle V) < \alpha$. There is a sequence (G_k) in $\mathcal{R}$, with $A \triangle V \subset \bigcup G_k$, such that $m_p(G_k) < \alpha$ for al k. As $N_{m,p}(x) > \alpha$, we have that $x \notin \bigcup G_k$ and so $x \notin A \triangle V$, which implies that $x \in V$. Moreover, $V \subset A \bigcup (V \setminus A)$ and thus

$$N_{m,p}(x) \le m_p(V) = m_p'^{\sigma}(V) \le \max\{m_p'^{\sigma}(A), m_p'^{\sigma}(V \setminus A)\} < \alpha,$$

a contradiction.

Lemma 4.6 *$\hat{\mathcal{R}}_m$ is a σ-algebra and m'^{σ} is σ-additive.*

Proof : We prove first that m'^{σ} is strongly additive. Indeed, let (A_n) be a sequence of pairwise disjoint members of $\hat{\mathcal{R}}_m$, $p \in cs(E)$ and $\epsilon > 0$. For each n, there exists $V_n \in \mathcal{R}$ with $m_p'^{\sigma}(A_n \triangle V_n) < \epsilon$. Let $W_1 = V_1$, $W_{n+1} = V_{n+1} \setminus \bigcup_{k=1}^{n} V_k$. Then

$$A_{n+1} \triangle W_{n+1} \subset \bigcup_{k=1}^{n+1} A_k \triangle V_k$$

and so $m_p'^{\sigma}(A_{n+1} \triangle W_{n+1}) < \epsilon$. The sets W_n, $n = 1, 2, \ldots$, are pairwise disjoint. Since m is strongly additive, there exists n_o such that $m_p(W_n) < \epsilon$ for $n \geq n_o$. Now, for $n \geq n_o$, we have $A_n = (A_n \cap W_n) \bigcup (A_n \setminus W_n)$ and hence

$$\begin{aligned} m_p'^{\sigma}(A_n) &= \max\{m_p'^{\sigma}(A_n \cap W_n), m_p'^{\sigma}(A_n \setminus W_n)\} \\ &\leq \max\{m_p'^{\sigma}(W_n), \hat{m}_p(A_n \triangle W_n)\} \\ &= \max\{m_p(W_n), \hat{m}_p(A_n \triangle W_n)\} < \epsilon. \end{aligned}$$

This proves that m'^{σ} is strongly additive. Next we show that $\hat{\mathcal{R}}_m$ is a σ-algebra. In fact, let (A_n) be a sequence in $\hat{\mathcal{R}}_m$ and $A = \bigcup A_n$. We need to show that $A \in \hat{\mathcal{R}}_m$. We may assume that the sets A_n are pairwise disjoint. Let $p \in cs(E)$ and $\epsilon > 0$. For each n there exists $V_n \in \mathcal{R}$ such that $\hat{m}_p(A_n \triangle V_n) < \epsilon$. Let $W_1 = V_1$, $W_{n+1} = V_{n+1} \setminus \bigcup_{k=1}^{n} V_k$. Then $\hat{m}_p(A_{n+1} \setminus W_{n+1}) < \epsilon$. Since the sets W_n are pairwise disjoint, there exists n such that $m_p(W_n) < \epsilon$ for all $k > n$. Now

$$A \triangle \left(\bigcup_{k=1}^{n} W_k \right) \subset \left[A \triangle \left(\bigcup_{1}^{\infty} W_k \right) \right] \bigcup \left[\bigcup_{k>n} W_k \right].$$

For each k there is a sequence $(B_{ki})_{i=1}^{\infty})$ in $\mathcal{R}$, such that $A_k \triangle W_k \subset \bigcup_i B_{ki}$ and $m_p(B_{ki}) < \epsilon$. If $G = A \triangle \left(\bigcup_{k=1}^{\infty} W_k \right)$, then

$$G \subset \bigcup_k A_k \triangle W_k \subset \bigcup_{k,i} B_{ki}$$

and hence $\hat{m}_p(G) \leq \sup_{k,i} m_p(B_{ki}) \leq \epsilon$. Also, for $F = \bigcup_{k>n} W_k$, we have

$$\hat{m}_p(F) \leq \sup_{k>n} m_p(W_k) \leq \epsilon.$$

Thus

$$\hat{m}_p(A \triangle (\cup_{k=1}^{n} W_k)) \leq \max\{\hat{m}_p(F), \hat{m}_p(G)\} \leq \epsilon,$$

which proves that $A \in \hat{\mathcal{R}}_m$.

Finally, m'^{σ} is σ-additive. In fact, let $(A_n) \subset \hat{\mathcal{R}}_m$, $A_n \downarrow \emptyset$. Let $B_n = A_n \setminus A_{n+1}$. The sets B_n are pairwise disjoint. Since m'^{σ} is strongly additive, there exists

N such that $m_p^{'\sigma}(B_n) = \hat{m}_p(B_n) < \epsilon$ for all $n \geq N$. Let $n \geq N$. Then $A_n = \bigcup_{k \geq n} B_k$. For each $k \geq n$, there is a sequence $(G_{ki})_{i=1}^{\infty}$ in $\mathcal{R}$ such that $B_k \subset \bigcup_i \bar{B}_{ki}$ and $m_p(B_{ki}) < \epsilon$. Now $A_n \subset \bigcup_{k \geq n} \cup_i B_{ki}$ and so

$$\hat{m}_p(A_n) \leq \sup_{k \geq n} \sup_i m_p(B_{ki}) \leq \epsilon.$$

This proves that $m_p^{'\sigma}(A_n) \to 0$ and so $m^{'\sigma}$ is σ-additive. This completes the proof.

Combining the preceding Lemmas, we get the following extension

Theorem 4.7 *Let $m \in M_\sigma(\mathcal{R}, E)$ be strongly additive. If $\mathcal{R}^\sigma$ is the σ-algebra generated by $\mathcal{R}$, then there exists a unique extension $m^\sigma \in M_\sigma(\mathcal{R}^\sigma, E)$ of m. Moreover $N_{m,p} = N_{m^\sigma,p}$.*

Proof : Since $\hat{\mathcal{R}}_m$ is a σ-algebra, it follows that $\mathcal{R}^\sigma$ is contained in $\hat{\mathcal{R}}_m$. Thus the restrtiction m^σ of $m^{'\sigma}$ to $\mathcal{R}^\sigma$ is a σ-additive extension of m. To prove the uniqueness, let $\mu \in M_\sigma(\mathcal{R}^\sigma, E)$ be an extension of m and let

$$\mathcal{F} = \{A \in \mathcal{R}^\sigma : \mu(A) = m^\sigma(A)\}.$$

It is easy to see that, if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$. The family $\mathcal{F}$ is a monotone class. Indeed, let (A_n) be a sequence in $\mathcal{F}$ with $A_n \downarrow A$. Then

$$\mu(A) = \lim \mu(A_n) = \lim m^\sigma(A_n) = m^\sigma(A).$$

Similarly, if $(B_n) \subset \mathcal{F}$ and $B_n \uparrow B$, then $m^\sigma(B) = \mu(B)$. Hence $\mathcal{F}$ is a monotone class. Since $\mathcal{R}^\sigma$ is the monotone class generated by $\mathcal{R}$ (by [6], Theorem B on p. 27), it follows that $\mathcal{F} = \mathcal{R}^\sigma$ and so $\mu = m^\sigma$. The equality $N_{m,p} = N_{m^\sigma,p}$ is a consequence of Lemma 4.5.

Theorem 4.8 *For $A \in \mathcal{R}^\sigma$ and $p \in cs(E)$, we have $m_p^\sigma(A) = m_p^{'\sigma}(A)$.*

Proof : It is clear that $m_p^\sigma(A) \leq m_p^{'\sigma}(A)$. On the other hand, let $B \in \hat{\mathcal{R}}_m$, $B \subset A$. There exists a net (V_δ) in $\mathcal{R}$ converging to B with respect to the uniformity $\mathcal{U}_m^\sigma$. But then $V_\delta \cap A \to B \cap A = B$. Hence

$$p(m^{'\sigma}(B)) = \lim_\delta p(m^{'\sigma}(A \cap V_\delta)) = \lim p(m^\sigma(A \cap V_\delta)).$$

Since $p(m^\sigma(A \cap V_\delta)) \leq m_p^\sigma(A)$, it follows that $p(m^{'\sigma}(B)) \leq m_p^\sigma(A)$, which proves that $m_p^\sigma(A) \geq m_p^{'\sigma}(A)$. This completes the proof.

Theorem 4.9 *Let $m \in M_\sigma(\mathcal{R}, E)$ and $\mu \in M_\sigma(\mathcal{R})$ both be strongly additive and suppose that $m \ll \mu$. Then $m^\sigma \ll \mu^\sigma$.*

Proof : Let $p \in cs(E)$ and $\epsilon > 0$. There exists a $\delta > 0$ such that, for $A \in \mathcal{R}$, if $|\mu|(A) < \delta$, then $m_p(A) < \epsilon$. Assume now $A \in \mathcal{R}^\sigma$ and $|\mu^\sigma|(A) < \delta$. There is a sequence (V_n) in $\mathcal{R}$ such that $A \subset \bigcup V_n$ and $|\mu|(V_n) < \delta$ for every n. But then $m_p^\sigma(V_n) = m_p(V_n) < \epsilon$, for every n, and so

$$m_p^\sigma(A) \leq \sup_n m_p^\sigma(V_n) \leq \epsilon.$$

This clearly completes the proof.

Theorem 4.10 $\hat{\mathcal{R}}_m = \mathcal{R}_m^\sigma$ *and* $m'^\sigma = \overline{m^\sigma}$.

Proof : Let $A \in \hat{\mathcal{R}}_m$. Given $p \in cs(E)$ and $\epsilon > 0$, there exists $V \in \mathcal{R}$ such that $\hat{m}_p(A \triangle V) < \epsilon$. Next, there is a sequence (G_n) in $\mathcal{R}$ such that $A \triangle V \subset G = \bigcup G_n$ and $m_p(G_n) < \epsilon$ for all n. Then $G \in \mathcal{R}^\sigma$. If $B = V \cap G^c$ and $F = V \cup G$, then $B \subset A \subset F$ and $F \setminus B = G$. Moreover

$$m_p^\sigma(G) = \sup_n m_p^\sigma(G_n) = \sup_n m_p(G_n) \leq \epsilon.$$

This proves that $A \in \mathcal{R}_m^\sigma$. Moreover, if $A_1 = B$, $A_2 = G$ and $A_3 = F^c$, then $\{A_1, A_2, A_3\}$ is an $\mathcal{R}^\sigma$-partition of X and, for $f = \chi_A$, we have that

$$|f(x) - f(y)| \cdot m_p^\sigma(A_k) \leq \epsilon,$$

if $x, y \in A_k$. If $x_k \in A_k$, then

$$\epsilon \geq p\left(\int f\, dm^\sigma - \sum_{k=1}^{3} f(x_k) m^\sigma(A_k)\right) = p\left(\int f\, dm^\sigma - m^\sigma(B) - f(x_2) m^\sigma(G)\right).$$

But

$$p(m'^\sigma(A) - m^\sigma(B)) = p(m'^\sigma(A \setminus B)) \leq m_p'^\sigma(A \setminus B) \leq m_p'^\sigma(F \setminus B) = m_p^\sigma(G) \leq \epsilon$$

and $p(m^\sigma(G)) \leq m_p^\sigma(G) \leq \epsilon$. Thus

$$p\left(\int f\, dm^\sigma - m'^\sigma(A)\right) \leq \epsilon.$$

It follows that

$$m'^\sigma(A) = \int f\, dm^\sigma = \overline{m^\sigma}(A).$$

Conversely, let $A \in \mathcal{R}_m^\sigma$. There are $B, F \in \mathcal{R}^\sigma$, with $B \subset A \subset F$ and $m_p^\sigma(F \setminus B) < \epsilon$. Now $A \triangle B \subset F \setminus B$ and

$$\hat{m}_p(A \triangle B) \leq \hat{m}_p(F \setminus B) = m_p^\sigma(F \setminus B) < \epsilon,$$

which proves that A is in the closure of $\mathcal{R}^\sigma$ with respect to the uniformity $\mathcal{U}_m^\sigma$ and hence $A \in \hat{\mathcal{R}}_m$. This completes the proof.

5 Extensions of τ-Additive Measures

Throughout this section, unless it is stated explicitly otherwise, m will be an element of $M_\tau(\mathcal{R}, E)$ which is strongly additive.
For $p \in cs(E)$, we define

$$\check{m}_p : P(X) \to \mathbb{R}, \quad \check{m}_p(A) = \inf \sup_i m_p(G_i),$$

where the infimum is taken over the collection of all families $(G_i)_{i \in I}$ of members of $\mathcal{R}$ which cover A. It is easy to show that

$$\check{m}_p(A \cup B) = \max\{\check{m}_p(A), \check{m}_p(B)\}.$$

Let

$$\check{d}_p : P(X) \times P(X) \to \mathbb{R}, \quad \check{d}_p(A, B) = \check{m}_p(A \triangle B).$$

Then $\check{d}_p$ is an ultrapseudometric on $P(X)$ and $\check{d}_p \le \hat{d}_p$. Let $\mathcal{U}_m^\tau$ be the uniformity induced by the pseudometrics $\check{d}_p$, $p \in cs(E)$. Then $\mathcal{U}_m^\tau$ is coarser than $\mathcal{U}_m^\sigma$. If $\check{\mathcal{R}}_m$ is the closure of $\mathcal{R}$ in $P(X)$ with respect to the topology induced by $\mathcal{R}_m^\tau$, , then $\hat{\mathcal{R}}_m^\sigma \subset \mathcal{R}_m^\tau$.

Lemma 5.1 *For $A \in \mathcal{R}$, we have that $m_p(A) = \check{m}_p(A)$.*

Proof : Clearly $m_p(A) \ge \check{m}_p(A)$. On the other hand, if (G_i) is a family of members of $\mathcal{R}$ covering A, then $m_p(A) \le \sup_i m_p(G_i)$, since m is τ-additive, which implies that $m_p(A) \le \check{m}_p(A)$.

Now for the map $m : \mathcal{R} \to E$, we have

$$p(m(A) - m(B)) \le m_p(A \triangle B) = \check{m}_p A \triangle B).$$

Thus m is uniformly continuous for the uniformity induced on $\mathcal{R}$ by $\mathcal{U}_m^\tau$. Hence, there exists a unique uniformly continuous extension $m'^\tau : \check{\mathcal{R}}_m \to E$.
The proofs of the following two Lemmas are analogous to the ones of Lemmas 4.2 and 4.3, respectively.

Lemma 5.2 *$\check{\mathcal{R}}_m$ is an algebra of subsets of X and $m'^\tau \in M(\check{\mathcal{R}}_m, E)$.*

Lemma 5.3 *1. For A, B subsets of X, we have*

$$|\check{m}_p(A) - \check{m}_p(B)| \le \check{m}_p(A \triangle B).$$

2. If $A, B \in \check{\mathcal{R}}_m$, then

$$|m_p'^\tau(A) - m_p'^\tau(B)| \le m_p'^\tau(A \triangle B).$$

Lemma 5.4 *1. For $A \in \mathcal{R}$, we have $m_p(A) = \check{m}_p(A)$.*
2. If $A \in \check{\mathcal{R}}_m$, then $\check{m}_p(A) = m_p^{'\tau}(A)$.

Proof : 1. Let $A \in \mathcal{R}$. By Lemma 5.1 we have $m_p(A) = \check{m}_p(A)$. Clearly $m_p^{'\tau}(A) \geq m_p(A)$. On the other hand, if $B \in \check{\mathcal{R}}_m$ is contained in A, then there exists a net (V_δ) in $\mathcal{R}$ which converges to B for the uniformity $\mathcal{U}_m^\tau$. But then $V_\delta \cap A \to B \cap A = B$ and so

$$p(m^{'\tau}(B)) = \lim p(m(V_\delta \cap A)) \leq m_p(A).$$

This proves that $m_p^{'\tau}(A) \leq m_p(A) \leq m_p^{'\tau}(A)$.
2. Let $A \in \check{\mathcal{R}}_m$. First we show that $m_p^{'\tau}(A) \leq \check{m}_p(A)$. Indeed, let $B \in \check{\mathcal{R}}_m$ be contained in A. There exists a net (W_δ) in $\mathcal{R}$ converging to B for the uniformity $\mathcal{U}_m^\tau$. Then $W_\delta \cap A \to B \cap A = B$ and so

$$p(m^{'\tau}(B)) = \lim p(m^{'\tau}(W_\delta \cap A)).$$

But

$$p(m^{'\tau}(W_\delta \cap A)) \leq m_p^{'\tau}(W_\delta) = \check{m}_p(W_\delta) \to \check{m}_p(B) \leq \check{m}_p(A)$$

and hence $p(m^{'\tau}(B)) \leq \check{m}_p(A)$, which proves that $\check{m}_p(A) \geq m_p^{'\tau}(A)$. Since $A \in \check{\mathcal{R}}_m$, there exists a net (V_δ) in $\mathcal{R}$ converging to A. Then

$$|\check{m}_p(V_\delta) - m_p^{'\tau}(A)| = |m_p^{'\tau}(V_\delta) - m_p^{'\tau}(A)| \leq m_p^{'\tau}(V_\delta \triangle A) \leq \check{m}_p(V_\delta \triangle A) \to 0,$$

which implies that $m_p^{'\tau}(A) = \lim \check{m}_p(V_\delta)$. Also, by the preceding Lemma, $\check{m}_p(V_\delta) \to \check{m}_p(A)$. Hence $\check{m}_p(A) = m_p^{'\tau}(A)$. This completes the proof.

The proof of the next Lemma is analogous to the one of Lemma 4.5.

Lemma 5.5 $N_{m,p} = N_{m^{'\tau},p}$.

Lemma 5.6 *$\check{\mathcal{R}}_m$ is a σ-algebra which contains the σ-algebra $\mathcal{R}^{bo}$ of all $\tau_\mathcal{R}$-Borel sets. Moreover, $m^{'\tau}$ is τ-additive.*

Proof : **Claim I.** For each family $(A_i)_{i\in I}$ of subsets of X and $A = \bigcup A_i$, we have $\check{m}_p(A) = \sup_i \check{m}_p(A_i) = d$. In fact, let $\alpha > d$. For each i, there exists a family $\mathcal{F}_i$ of members of $\mathcal{R}$ such that $A_i \subset \bigcup \mathcal{F}_i$ and $m_p(B) < \alpha$ for every $B \in \mathcal{F}_i$. If $\mathcal{F} = \bigcup_i \mathcal{F}_i$, then $A \subset \bigcup \mathcal{F}$ and $m_p(B) < \alpha$, for each $B \in \mathcal{F}$, which implies that $\check{m}_p(A) \leq \alpha$. It follows that $\check{m}_p(A) \leq d \leq \check{m}_p(A)$.
Claim II . $m^{'\tau}$ is strongly additive. Indeed, let (A_n) be a sequence of pairwise disjoint members of $\check{\mathcal{R}}_m$ and let $p \in cs(E)$ and $\epsilon > 0$. For each n, there exists $V_n \in \mathcal{R}$ such that $\check{m}_p(A_n \triangle V_n) < \epsilon$. Let $W_1 = V_1$ and $W_{n+1} = V_{n+1} \setminus \bigcup_{k=1}^n V_k$. Then $\check{m}_p(A_{n+1} \triangle W_{n+1}) < \epsilon$. The sets W_n are pairwise disjoint. Since m is

strongly additive, there exists n_o such that $m_p(W_n) < \epsilon$ for all $n \geq n_o$. Now, for $n \geq n_o$, we have $A_n = (A_n \cap W_n) \bigcup (A_n \setminus W_n)$ and hence

$$\begin{aligned} m_p^{'\tau}(A_n) &\leq \max\{m_p^{'\tau}(A_n \cap W_n), m_p^{'\tau}(A_n \setminus W_n)\} \\ &\leq \max\{m_p^{'\tau}(W_n), m_p^{'\tau}(A_n \triangle W_n)\} < \epsilon, \end{aligned}$$

and the claim follows.

Claim III . $\breve{\mathcal{R}}_m$ is a σ-algebra. In fact, let (A_n) be a sequence in $\breve{\mathcal{R}}_m$ and $A = \bigcup A_n$. We may assume that the sets A_n are pairwise disjoint. Given $p \in cs(E)$ and $\epsilon > 0$, there is a sequence (W_n) of pairwise disjoint members of $\mathcal{R}$ such that $\breve{m}_p(A_n \triangle W_n) < \epsilon$ for every n. As the sets W_n are pairwise disjoint, there exists n such that $m_p(W_k) < \epsilon$ for all $k > n$. Now

$$\begin{aligned} A\triangle \left(\bigcup_{k=1}^{n} W_k\right) &\subset \left[A\triangle \left(\bigcup_{1}^{\infty} W_k\right)\right] \bigcup \left[\bigcup_{k>n} W_k\right] \\ &\subset \left[\bigcup_{k=1}^{\infty} A_k \triangle W_k\right] \bigcup \left[\bigcup_{k>n} W_k\right]. \end{aligned}$$

In view of Claim I, we get that $\breve{m}_p \left(A\triangle \left(\bigcup_{k=1}^{n} W_k\right)\right) \leq \epsilon$. This proves that $A \in \breve{\mathcal{R}}_m$.

Claim IV . Every $\tau_{\mathcal{R}}$-closed set is in $\breve{\mathcal{R}}_m$ and so $\mathcal{R}^{bo}$ is contained in $\breve{\mathcal{R}}_m$. Indeed, if A is $\tau_{\mathcal{R}}$-closed, then the exists a decreasing net (V_δ) in $\mathcal{R}$ with $A = \bigcap V_\delta$. Since m is strongly additive, the exists δ_o such that $m_p(V_{\delta_o} \setminus V_\delta) \leq \epsilon$ for all $\delta \geq \delta_o$. Since $A^c = \bigcup_{\delta \geq \delta_o} V_\delta^c$, we get that

$$A^c \cap V_{\delta_o} = \bigcup_{\delta \geq \delta_o} V_{\delta_o} \setminus V_\delta \quad \text{and so} \quad \breve{m}_p(A^c \cap V_{\delta_o}) \leq \epsilon.$$

But $A^c \cap V_{\delta_o} = A \triangle V_{\delta_o}$. This proves that $A \in \breve{\mathcal{R}}_m$ and the claim follows.

Claim V . $m^{'\tau}$ is τ-additive. In fact, let $A_\delta)$ be a net in $\breve{\mathcal{R}}_m$ with $A_\delta \downarrow \emptyset$. Since $m^{'\tau}$ is strongly additive, given $p \in cs(E)$ and $\epsilon > 0$, there exists δ_o such that $m_p^{'\tau}(A_{\delta_o} \setminus A_\delta) < \epsilon$ for all $\delta \geq \delta_o$. As $A_{\delta_o} = \bigcup_{\delta \geq \delta_o} A_{\delta_o} \setminus A_\delta$, we get that

$$m_p^{'\tau}(A_{\delta_o}) = \breve{m}_p(A_{\delta_o}) = \sup_{\delta \geq \delta_o} \breve{m}_p(A_{\delta_o} \setminus A_\delta) = \sup_{\delta \geq \delta_o} m_p^{'\tau}(A_{\delta_o} \setminus A_\delta) \leq \epsilon.$$

Hence $\lim m_p^{'\tau}(A_\delta) = 0$ and so $m^{'\tau}$ is τ-additive.

Theorem 5.7 *If m^τ is the restriction of $m^{'\tau}$ to $\mathcal{R}^{bo}$, then $m^\tau \in M_\tau(\mathcal{R}^{bo}, E)$ is the unique τ-additive extension of m to $\mathcal{R}^{bo}$. Moreover $m^\tau|_{\mathcal{R}^\sigma} = m^\sigma$.*

Proof : Assume that $\mu \in M_\tau(\mathcal{R}^{bo}, E)$ is an extension of m. We first show that $\mu(A) = m^\tau(A)$ for each $\tau_{\mathcal{R}}$-closed set A. Indeed, there exists a decreasing net

(V_δ) in $\mathcal{R}$ with $A = \bigcap V_\delta$. Let $B_\delta = A^c \cap V_\delta$. Then $B_\delta \downarrow \emptyset$ and so $m^\tau(B_\delta) \to 0$ and $\mu(B_\delta) \to 0$. Since $V_\delta = A \cup B\delta$, we have that

$$\mu(A) - m^\tau(A) = m^\tau(B_\delta) - \mu(B_\delta) \to 0,$$

and hence $\mu(A) = m^\tau(A)$. Also $\mu(B) = m^\tau(B)$ for each $\tau_{\mathcal{R}}$-open set B since $\mu(X) = m^\tau(X)$. For A a $\tau_{\mathcal{R}}$-open set and B a $\tau_{\mathcal{R}}$-closed set , we have that

$$\mu(A \cap B) = \mu(A) - \mu(A \cap B^c) = m^\tau(A) - m^\tau(A \cap B^c) = m^\tau(A \cap B).$$

It is easy to show that the family $\mathcal{F}$ of all finite unions of sets of the form $A \cap B$, where A is $\tau_{\mathcal{R}}$-open and B $\tau_{\mathcal{R}}$-closed, is an algebra. Moreover, every member of $\mathcal{F}$ is a finite union of pairwise disjoint membres of $\mathcal{F}$. Thus $\mu(G) = m^\tau(G)$ for every member G of $\mathcal{F}$. It is clear that $\mathcal{R}^{bo}$ coincides with the σ-algebra generated by $\mathcal{F}$. As $\mathcal{F}$ is an algebra, $\mathcal{R}^{bo}$ coincides with the monotone class generated by $\mathcal{F}$ (by Halmos [6], Theorem B on p. 27). The class $\mathcal{F}_1$, of all members A of $\mathcal{R}^{bo}$ for which $\mu(A) = m^\tau(A)$, is monotone. It follows that $\mu = m^\tau$ on $\mathcal{R}^{bo}$. Finaly, if $m_1 = m^\tau|_{\mathcal{R}^\sigma}$, then m_1 is a σ-additive extension of m and thus $m_1 = m^\sigma$ by the uniqunees part of Theorem 4.7. This completes the proof.

The proof of the following Theorem is analogous to the one of Theorem 4.8.

Theorem 5.8 *For $A \in \mathcal{R}^{bo}$ and $p \in cs(E)$, we have $m_p^\tau(A) = m_p'^\tau(A)$.*

Theorem 5.9 *For $A \in \mathcal{R}^\sigma$ and $p \in cs(E)$, we have $m_p^\sigma(A) = m_p^\tau(A)$.*

Proof : There exists a net (V_δ) in $\mathcal{R}$ which converges to A with respect to the uniformity $\mathcal{U}_m^\sigma$. Since $\mathcal{U}_m^\tau$ is coarser than $\mathcal{U}_m^\sigma$, (V_δ) converges to A with respect to $\mathcal{U}_m^\tau$. Now

$$m_p^\sigma(V_\delta) \to m_p^\sigma(A) \quad \text{and} \quad m_p^\tau(V_\delta) \to m_p^\tau(A).$$

Since $m_p^\sigma(V_\delta) = m_p^\tau(V_\delta) = m_p(V_\delta)$, the Theorem follows.

Theorem 5.10 *$\check{\mathcal{R}}_m = \mathcal{R}_m^{bo}$ and $m'^\tau = \overline{m^\tau}$.*

Proof : Let $A \in \check{\mathcal{R}}_m$. Given $p \in cs(E)$ and $\epsilon > 0$, there exists $V \in \mathcal{R}$ with $\check{m}_p(A \triangle V) < \epsilon$. Let (G_i) be a family in $\mathcal{R}$ such that $A \triangle V \subset G = \bigcup G_i$ and $m_p(G_i) < \epsilon$ for every i. If $B = V \cap G^c$ and $F = V \cup G$, then $B \subset A \subset F$. Moreover $F \setminus B = G$ and

$$m_p^\tau(G) = \sup_i m_p^\tau(G_i) = \sup_i m_p(G_i) \leq \epsilon.$$

This proves that $A \in \mathcal{R}_m^{bo}$. Moreover, if $B_1 = B, B_2 = G$ and $B_3 = F^c$, then $\{B_1, B_2, B_3\}$ is an $\mathcal{R}^{bo}$-partition of X and, for $f = \chi_A$, we have

$$|f(x) - f(y)| \cdot m_p^\tau(B_k) \leq \epsilon$$

if $x, y \in B_k$. If $x_k \in B_k$, then

$$\begin{aligned}\epsilon \geq p\left(\int f\,dm^\tau - \sum_{k=1}^{3} f(x_k) m^\tau(B_k)\right) \\ = p\left(\overline{m^\tau}(A) - m^\tau(B) - f(x_2)m^\tau(G)\right).\end{aligned}$$

But

$$p(m'^\tau(A) - m^\tau(B)) = p(m'^\tau(A \setminus B)) \leq \check{m}_p(G) = m_p^\tau(G) \leq \epsilon$$

and $p(m^\tau(G)) \leq m_p^\tau(G) \leq \epsilon$. Thus

$$p(\overline{m^\tau}(A) - m'^\tau(A)) \leq \epsilon.$$

This proves that $\overline{m^\tau}(A) = m'^\tau(A)$.
Conversely, let $A \in \mathcal{R}_m^{bo}$. Then there are $B, F \in \mathcal{R}^{bo}$ such that $B \subset A \subset F$ and $m_p^\tau(F \setminus B) < \epsilon$. Now $A \triangle B \subset F \setminus B$ and

$$\check{m}_p(A \triangle B) \leq \check{m}_p(F \setminus B) = m_p^\tau(F \setminus B) < \epsilon,$$

which proves that A is in the closure of $\mathcal{R}^{bo}$ in P(X) with respect to the uniformity $\mathcal{U}_m^\tau$. Hence $A \in \check{\mathcal{R}}_m$. This completes the proof.

The proof of the following Theorem is analogous to the one of Theorem 4.9.

Theorem 5.11 *Let $m \in M_\tau(\mathcal{R}, E)$ and $\mu \in M_\tau(\mathcal{R})$ be both strongly additive. If $m \ll \mu$, then $m^\tau \ll \mu^\tau$.*

Theorem 5.12 *Let $m \in M(\mathcal{R}, E)$ be strongly additive and let $f \in \mathbb{K}^X$ be m-integrable. Then:*

1. *If m is σ-additive, the f is m^σ-integrable and $\int f\,dm = \int f\,dm^\sigma$.*
2. *If m is τ-additive, then f is m^τ-integrable and $\int f\,dm = \int f\,dm^\tau$.*

Proof : 1. Assume that m is σ-additive and let $p \in cs(E)$ and $\epsilon > 0$. Since f is m-integrable, there exists an $\mathcal{R}$-partition $\{A_1, A_2 \ldots, A_n\}$ of X such that

$$|f(x) - f(y)| \cdot m_p(A_k) < \epsilon$$

if $x, y \in A_k$. Since $m_p^\sigma(A_k) = m_p(A_k)$, it follows that f is m^σ-integrable. Moreover, if $x_k \in A_k$, then

$$p\left(\int f\,dm - \sum_{k=1}^{n} f(x_k)m(A_k)\right) \leq \epsilon \quad \text{and} \quad p\left(\int f\,dm^\sigma - \sum_{k=1}^{n} f(x_k)m^\sigma(A_k)\right) \leq \epsilon,$$

which implies that $p\left(\int f\,dm - \int f\,dm^\sigma\right) \leq \epsilon$. It follows that $\int f\,dm = \int f\,dm^\sigma$ since E is Hausdorff.
2. The proof is analogous to that of (1).

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ON THE EXISTENCE OF POSITIVE SOLUTIONS ON THE HALF-LINE TO NONLINEAR TWO-DIMENSIONAL DELAY DIFFERENTIAL SYSTEMS

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ABSTRACT. The paper is concerned with a boundary value problem on the half-line to nonlinear two-dimensional delay differential systems with positive delays. A theorem is established, which provides sufficient conditions for the existence of positive solutions. The application of this theorem to the special case of second order nonlinear delay differential equations is given. Also, the application of the theorem to two-dimensional Emden-Fowler type delay differential systems with constant delays is presented. Moreover, some general examples demonstrating the applicability of the theorem are included.

1. INTRODUCTION

Recently, the author [31] established sufficient conditions for the existence of positive increasing solutions of a boundary value problem on the half-line to second order nonlinear delay differential equations *with positive delays.* The assumption that the delays are positive is essential to the approach in [31], and hence the results given in [31] cannot be applied to the corresponding boundary value problem for second order nonlinear *ordinary* differential equations. An old idea that appeared in the author's paper [30] plays a crucial role in the study in [31]. (Grains of this idea were presented in the old paper by Lovelady [19].)

Also, recently, the author [32] studied the problem of the existence of solutions and of the existence and uniqueness of solutions of a boundary value problem on the half-line to nonlinear two-dimensional delay differential systems. The methods applied in [32] are based on the use of the Schauder-Tikhonov theorem and the Banach contraction principle. The results obtained in [32] include, as special cases, those given by Mavridis, the present author and Tsamatos [20] for second order nonlinear delay differential equations.

The work in [31, 32] is closely related to the work in the papers by Mavridis, the author and Tsamatos [20, 21] and, in a sense, to the work in the paper by Agarwal, the author and Tsamatos [2].

The present paper is essentially motivated by the recent work in [31] (and, in a sense, by the recent work in [32]). In this paper, a boundary value problem on the half-line to nonlinear two-dimensional delay differential systems *with positive delays* is considered, and sufficient conditions are given that guarantee the exisence of positive solutions. The results obtained are not applicable to the corresponding boundary value problem for nonlinear two-dimensional *ordinary* differential

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systems. The results given in [31] can be derived, as special consequences, from the ones established here, by reducing a second order nonlinear delay differential equation to a nonlinear two-dimensional delay differential system of a special form. The techniques applied in this paper are originated in the ones in [31]; also, some elements of the methods used in [32] are succesfully employed in the present paper.

Consider the nonlinear two-dimensional delay differential system

$$x'(t) = g\left(t, y(t)\right), \qquad y'(t) = -f(t, x\left(t - T_1(t)\right), ..., x\left(t - T_m(t)\right)), \tag{1.1}$$

where m *is a positive integer*, g *is a continuous real-valued function on* $[0,\infty) \times \mathbb{R}$, f *is a continuous real-valued function on* $[0,\infty) \times \mathbb{R}^m$, *and* T_j $(j = 1, ..., m)$ *are positive continuous real-valued functions on the interval* $[0,\infty)$ *with*

$$\lim_{t \to \infty} \left(t - T_j(t)\right) = \infty \quad (j = 1, ..., m).$$

Let us consider the positive real number τ defined by

$$\tau = - \min_{j=1,\dots,m} \min_{t \geq 0} \left(t - T_j(t)\right).$$

Our interest will be concentrated on *global* solutions of the delay differential system (1.1), i.e., on solutions of (1.1) on the *whole* interval $[0,\infty)$. By a *solution on* $[0,\infty)$ of (1.1), we mean a pair of two continuous real-valued functions x and y defined on the intervals $[-\tau,\infty)$ and $[0,\infty)$, respectively, which are continuously differentiable on $[0,\infty)$ and satisfy (1.1) for all $t \geq 0$.

Together with the delay differential system (1.1), we specify an initial condition of the form

$$x(t) = \phi(t) \quad \text{for } -\tau \leq t \leq 0, \tag{1.2}$$

where the initial function ϕ *is a given continuous real-valued fucntion on the interval* $[-\tau, 0]$. Throughout the paper, *it will be assumed that*

$$\phi(0) = 0.$$

Moreover, along with (1.1), we impose the condition

$$\lim_{t \to \infty} y(t) = 0. \tag{1.3}$$

The delay differential system (1.1) together with the conditions (1.2) and (1.3) constitute a *boundary value problem* (*BVP*, for short) *on the half-line*. A solution on $[0,\infty)$ of (1.1) satisfying (1.2) and (1.3) is said to be a *solution* of the boundary value problem (1.1)$-$(1.3) or, more briefly, a *solution* of the BVP (1.1)$-$(1.3).

Proposition 1.1, below, provides a useful integral representation of the BVP (1.1)$-$(1.3), which will be used in proving the main result of the paper (and in proving a basic lemma needed in the proof of our main result). This proposition has been established by the author [32] for a more general boundary value problem on the half-line to more general nonlinear two-dimensional delay differential systems, in which, however, the delays are assumed to be bounded. But, as it is easy to see, the restriction of the boundedness of the delays is not needed for the validity of the proposition.

Proposition 1.1. *Let x and y be two continuous real-valued functions defined on the intervals $[-\tau,\infty)$ and $[0,\infty)$, respectively. Then (x, y) is a solution of the*

BVP (1.1)$-$(1.3) *if and only if*

$$x(t)=\begin{cases}\phi(t) & \text{for } -\tau\le t\le 0\\ \int_0^t g(s,y(s))ds & \text{for } t\ge 0\end{cases} \tag{1.4}$$

and

$$y(t)=\int_t^{\infty} f(s,x(s-T_1(s)),\ldots,x(s-T_m(s)))ds \quad \text{for } t\ge 0. \tag{1.5}$$

We are interested in studying the problem of the existence of solutions (x,y) of the BVP (1.1)$-$(1.3) with x being positive on $[-\tau,\infty)-\{0\}$. Therefore, in addition to the assumption that $\phi(0)=0$ posed previously, without mentioning it any further, *it will be supposed that*

$$\phi(t)>0 \quad \text{for } -\tau\le t<0.$$

The main result of this paper is Theorem 3.1, which will be stated and proved in Section 3. This theorem provides sufficient conditions for the BVP (1.1)$-$(1.3) to have at least one solution (x,y) such that x is positive on $(0,\infty)$ and y is positive on $[0,\infty)$. A crucial role in proving Theorem 3.1 plays Lemma 2.1, which will be established in Section 2; this lemma gives useful information about the solutions (x,y) of the BVP (1.1)$-$(1.3) with x being nonnegative on $(0,\infty)$. Section 4 is devoted to the application of Theorem 3.1 (as well as of Lemma 2.1) to the special case of second order nonlinear delay differential equations. Section 5 contains the application of Theorem 3.1 to (nonlinear) two-dimensional Emden-Fowler type differential systems with constant delays. Also, some general examples, which demonstrate the applicability of our main result, will be presented in Section 5.

The problem studied in the present paper is closely related to the general problem of deriving sufficient conditions for the existence of solutions with prescribed asymptotic behavior to second (or arbitrary) order nonlinear ordinary and delay differential equations. Among numerous articles dealing with this general problem, we choose to refer to the most recent papers [1$-$3, 6$-$9, 17, 18, 20$-$22, 24$-$29, 31, 33$-$37, 39]; we, also, refer to the old classical articles [13, 14], and to the paper [40].

On the other hand, several articles have appeared in the literature, which are concerned with the asymptotic behavior of solutions of nonlinear ordinary differential systems. See, for example, [12, 15, 16, 38]; in particular, see the monograph by Mirzov [23] and the references cited therein.

For the basic theory of delay differential equations and systems, the reader is referred to the books by Diekmann *et al.* [4], Driver [5], Hale [10], and Hale and Vertuyn Lunel [11].

2. A BASIC LEMMA

Here, we will establish the following basic lemma.

Lemma 2.1. *Assume that the function g is positive on $[0,\infty)\times(0,\infty)$, i.e.,*

$$g(t,z)>0 \quad \text{for all } t\ge 0 \text{ and } z>0. \tag{2.1}$$

Also, assume that the function f is positive on $[0,\infty)\times(0,\infty)^m$, i.e.,

$$f(t,w_1,...,w_m)>0 \quad \text{for all } t\geq 0 \text{ and } w_1>0,...,w_m>0. \tag{2.2}$$

Let (x,y) be a solution of the BVP (1.1)–(1.3) with x being nonnegative on the interval $(0,\infty)$. Then x is always positive on $(0,\infty)$; moreover, y is positive on $[0,\infty)$.

We notice here that, because of the continuity of g on $[0,\infty)\times[0,\infty)$, the hypothesis that g is positive on $[0,\infty)\times(0,\infty)$, i.e., that (2.1) holds, implies that *the function g is nonnegative on $[0,\infty)\times[0,\infty)$, i.e.,*

$$g(t,z)\geq 0 \quad \text{for all } t\geq 0 \text{ and } z\geq 0. \tag{2.3}$$

Similarly, as f is continuous on $[0,\infty)\times[0,\infty)^m$, the hypothesis that f is positive on $[0,\infty)\times(0,\infty)^m$, i.e., that (2.2) holds, guarantees that *the function f is nonnegative on $[0,\infty)\times[0,\infty)^m$, i.e.,*

$$f(t,w_1,...,w_m)\geq 0 \quad \text{for all } t\geq 0 \text{ and } w_1\geq 0,...,w_m\geq 0. \tag{2.4}$$

Now, we shall present an observation. *Assume that (2.1) holds, and let (x,y) be a solution of the BVP (1.1)–(1.3) such that y is positive on the interval $[0,\infty)$.* Then, from the first equation of (1.1), it follows that

$$x'(t)>0 \quad \text{for every } t\geq 0$$

and so x is strictly increasing on $[0,\infty)$. Hence, as $x(0)=\phi(0)=0$, *x is positive on $(0,\infty)$.*

Proof of Lemma 2.1. The proof will be accomplished by proving that y is positive on the interval $[0,\infty)$.

First of all, we see that x is nonnegative on the whole interval $[-\tau,\infty)$ and so we must have $x(t-T_j(t))\geq 0$ for $t\geq 0$ $(j=1,...,m)$. Consequently, by (2.4),

$$f(t,x(t-T_1(t)),...,x(t-T_m(t)))\geq 0 \quad \text{for every } t\geq 0. \tag{2.5}$$

Moreover, we observe that, by Proposition 1.1, the solution (x,y) satisfies (1.4) and (1.5).

Now, we will show that $y(0)>0$. To this end, by applying (1.5) for $t=0$, we get

$$y(0)=\int_0^\infty f(s,x(s-T_1(s)),...,x(s-T_m(s)))ds. \tag{2.6}$$

As $-\tau\leq -T_j(0)<0$ $(j=1,...,m)$, we have $x(-T_j(0))=\phi(-T_j(0))>0$ $(j=1,...,m)$. Thus, because of (2.2), we must have

$$f(0,x(-T_1(0)),...,x(-T_m(0)))>0,$$

i.e.,

$$f(t,x(t-T_1(t)),...,x(t-T_m(t)))|_{t=0}>0.$$

In view of (2.5) and the last inequality, it follows from (2.6) that $y(0)$ is always positive.

Next, we shall prove that y is positive on the interval $(0,\infty)$. Assume, for the sake of contradiction, that y is not necessarily positive on $(0,\infty)$. Then, as $y(0)>0$, we see that y has always zeros in the interval $(0,\infty)$. Let $t_0>0$ be the first zero of y in $(0,\infty)$; i.e., y is positive on $[0,t_0)$, and $y(t_0)=0$. In view of (2.1) and (2.3), it follows from the first equation of (1.1) that $x'(t)>0$ for $t\in[0,t_0)$, and

$x'(t_0) \geq 0$. Hence, x is strictly increasing on $[0, t_0)$ and x is increasing on $[0, t_0]$. Thus, as $x(0) = \phi(0) = 0$, we see that x is always positive on $(0, t_0]$. Furthermore, by taking into account the fact that $y(t_0) = 0$ and applying (1.5) for $t = t_0$, we obtain

$$\int_{t_0}^{\infty} f(s, x(s - T_1(s)), ..., x(s - T_m(s)))ds = 0.$$

So, because of (2.5), we must have

$$f(t, x(t - T_1(t)), ..., x(t - T_m(t))) = 0 \quad \text{for every } t \geq t_0. \tag{2.7}$$

By (2.7), the second equation of (1.1) gives $y'(t) = 0$ for all $t \geq t_0$, which means that y is constant on $[t_0, \infty)$. Hence, since $y(t_0) = 0$, we have $y(t) = 0$ for every $t \geq t_0$. So, by taking into account (2.3), from (1.4) we obtain, for each $t \geq t_0$,

$$x(t) = \int_0^{t_0} g(s, y(s))\, ds + \int_{t_0}^{t} g(s, y(s))\, ds = x(t_0) + \int_{t_0}^{t} g(s, 0)ds \geq x(t_0).$$

Thus, as $x(t_0) > 0$, we have $x(t) > 0$ for every $t \geq t_0$. Consequently, x is always positive on the interval $(0, \infty)$. Finally, by the assumption that $\lim_{t\to\infty} (t - T_j(t)) = \infty$ $(j = 1, ..., m)$, we can consider a point $t_1 > 0$ so that $t - T_j(t) > 0$ for all $t \geq t_1$ $(j = 1, ..., m)$. Then, as x is positive on $(0, \infty)$, we have $x(t - T_j(t)) > 0$ for every $t \geq t_1$ $(j = 1, ..., m)$. Therefore, by using (2.2), we find that

$$f(t, x(t - T_1(t)), ..., x(t - T_m(t))) > 0 \quad \text{for all } t \geq t_1,$$

which contradicts (2.7).

The proof of the lemma is complete.

3. THE MAIN RESULT

Our main result is the following theorem.

Theorem 3.1. *Let the assumptions of Lemma 2.1 be satisfied, i.e.,* (2.1) *and* (2.2) *hold. Moreover, assume that, for each $t \geq 0$, the function $g(t, \cdot)$ is increasing on $[0, \infty)$ in the sense that $g(t, z_1) \leq g(t, z_2)$ for any z_1, z_2 in $[0, \infty)$ with $z_1 \leq z_2$. Also, assume that, for each $t \geq 0$, the function $f(t, \cdot, ..., \cdot)$ is increasing on $[0, \infty)^m$ in the sense that $f(t, w_1, ..., w_m) \leq f(t, v_1, ..., v_m)$ for any $(w_1, ..., w_m)$, $(v_1, ..., v_m)$ in $[0, \infty)^m$ with $w_1 \leq v_1$, ..., $w_m \leq v_m$.*

Let there exists a real number $c > 0$ so that

$$\int_0^{\infty} f(t, \rho_1(t), ..., \rho_m(t))\, dt \leq c, \tag{3.1}$$

where, for each $j \in \{1, ..., m\}$, the function ρ_j depends on ϕ, c, g and is defined by

$$\rho_j(t) = \begin{cases} \phi(t - T_j(t)), & \text{if } 0 \leq t \leq T_j(t) \\ \int_0^{t - T_j(t)} g(s, c)ds, & \text{if } t \geq T_j(t). \end{cases} \tag{3.2}$$

(Clearly, ρ_j $(j = 1, ..., m)$ are nonnegative continuous real-valued functions on the interval $[0, \infty)$.) Then the BVP (1.1)–(1.3) *has at least one solution (x, y) such that*

$$0 < x(t) \leq \int_0^t g(s, c)ds \quad \textit{for every } t > 0 \tag{3.3}$$

and

$$0 < y(t) \leq c \quad \textit{for every } t \geq 0. \tag{3.4}$$

Proof. Let Y be the set of all continuous real-valued functions y defined on the interval $[0, \infty)$, that satisfy

$$0 \leq y(t) \leq c \quad \text{for every } t \geq 0. \tag{3.5}$$

For any function y in Y, let x denote the continuous real-valued function on the interval $[-\tau, \infty)$ defined by the formula (1.4). (Note that $\phi(0) = 0$.)

Consider an arbitrary function y in Y. Then, in view of (3.5), we can use (2.3) as well as the assumption that, for each $t \geq 0$, the function $g(t, \cdot)$ is increasing on $[0, \infty)$ to obtain

$$0 \leq g(t, y(t)) \leq g(t, c) \quad \text{for } t \geq 0.$$

This gives

$$0 \leq \int_0^t g(s, y(s))\, ds \leq \int_0^t g(s, c) ds \quad \text{for } t \geq 0,$$

which, by the definition of x by (1.4), is written as

$$0 \leq x(t) \leq \int_0^t g(s, c) ds \quad \text{for every } t \geq 0. \tag{3.6}$$

From (1.4) and (3.6) it follows that, for any $j \in \{1, ..., m\}$ and every $t \geq 0$,

$$\begin{cases} 0 \leq x(t - T_j(t)) = \phi(t - T_j(t)), & \text{if } 0 \leq t \leq T_j(t) \\ 0 \leq x(t - T_j(t)) \leq \int_0^{t - T_j(t)} g(s, c) ds, & \text{if } t \geq T_j(t). \end{cases}$$

(Note that $x(0) = \phi(0) = 0$.) Thus, by virtue of (3.2), we have

$$0 \leq x(t - T_j(t)) \leq \rho_j(t) \quad \text{for every } t \geq 0 \qquad (j = 1, ..., m).$$

Hence, by using (2.4) as well as the assumption that, for each $t \geq 0$, the function $f(t, \cdot, ..., \cdot)$ is increasing on $[0, \infty)^m$, we find that

$$0 \leq f(t, x(t - T_1(t)), ..., x(t - T_m(t))) \leq f(t, \rho_1(t), ..., \rho_m(t)) \quad \text{for } t \geq 0. \tag{3.7}$$

Taking into account (3.7), we obtain, for $t \geq 0$,

$$\begin{aligned} 0 &\leq \int_t^\infty f(s, x(s - T_1(s)), ..., x(s - T_m(s))) ds \\ &\leq \int_t^\infty f(s, \rho_1(s), ..., \rho_m(s)) ds \\ &\leq \int_0^\infty f(s, \rho_1(s), ..., \rho_m(s)) ds \end{aligned}$$

and consequently, because of hypothesis (3.1),

$$0 \leq \int_t^\infty f(s, x(s - T_1(s)), ..., x(s - T_m(s))) ds \leq c \quad \text{for every } t \geq 0. \tag{3.8}$$

As (3.8) holds true for all functions y in Y, we see that the formula

$$(My)(t) = \int_t^\infty f(s, x(s - T_1(s)), ..., x(s - T_m(s))) ds \quad \text{for } t \geq 0$$

makes sense for any y in Y, and defines a mapping M of Y into itself. We will show that the mapping M is increasing with respect to the usual pointwise ordering in Y. To this end, let us consider two arbitrary functions y and $\widetilde{y}$ in Y with $y \leq \widetilde{y}$, i.e., with $y(t) \leq \widetilde{y}(t)$ for $t \geq 0$. Let $\widetilde{x}$ denote the continuous real-valued function on $[-\tau, \infty)$ defined by the formula (1.4) with $\widetilde{x}$ instead of x and $\widetilde{y}$ in place of y, i.e.,

$$\widetilde{x}(t) = \begin{cases} \phi(t) & \text{for } -\tau \leq t \leq 0 \\ \int_0^t g(s, \widetilde{y}(s))\, ds & \text{for } t \geq 0. \end{cases} \tag{3.9}$$

As $0 \leq y(t) \leq \widetilde{y}(t)$ for $t \geq 0$, by using (2.3) as well as the assumption that, for each $t \geq 0$, the function $g(t, \cdot)$ is increasing on $[0, \infty)$, we get

$$0 \leq \int_0^t g(s, y(s))\, ds \leq \int_0^t g(s, \widetilde{y}(s))\, ds \quad \text{for } t \geq 0.$$

So, by taking into account the definitions of x and $\widetilde{x}$ by (1.4) and (3.9), respectively, we have

$$0 \leq x(t) \leq \widetilde{x}(t) \quad \text{for every } t \geq 0.$$

Thus, having in mind (1.4) and (3.9), we obtain, for each $j \in \{1, ..., m\}$ and every $t \geq 0$,

$$\begin{cases} 0 \leq x(t - T_j(t)) = \phi(t - T_j(t)) = \widetilde{x}(t - T_j(t)), & \text{if } 0 \leq t \leq T_j(t) \\ 0 \leq x(t - T_j(t)) \leq \widetilde{x}(t - T_j(t)), & \text{if } t \geq T_j(t). \end{cases}$$

Hence, by the assumption that, for each $t \geq 0$, the function $f(t, \cdot, ..., \cdot)$ is increasing on $[0, \infty)^m$, we derive

$$f(t, x(t - T_1(t)), ..., x(t - T_m(t))) \leq f(t, \widetilde{x}(t - T_1(t)), ..., \widetilde{x}(t - T_m(t)))$$

for all $t \geq 0$. This gives immediately

$$(My)(t) \leq (M\widetilde{y})(t) \quad \text{for every } t \geq 0,$$

i.e., $My \leq M\widetilde{y}$. Consequently, the mapping M is increasing.

Now, we define

$$y_0(t) = c \quad \text{for } t \geq 0$$

and

$$y_{n+1} = My_n \quad (n = 0, 1, ...).$$

As M is an increasing mapping of Y into itself, it is not difficult to see that $(y_n)_{n=0,1,...}$ is a decreasing sequence of functions in Y. Set

$$y = \lim_{n \to \infty} y_n \quad \text{pointwise on } [0, \infty).$$

Let x be defined by (1.4). Moreover, for any integer $n \geq 0$, let x_n denote the continuous real-valued function on $[-\tau, \infty)$ defined by (1.4) with x_n in place of x and y_n instead of y, i.e.,

$$x_n(t) = \begin{cases} \phi(t) & \text{for } -\tau \leq t \leq 0 \\ \int_0^t g(s, y_n(s))\, ds & \text{for } t \geq 0. \end{cases}$$

Then

$$x = \lim_{n \to \infty} x_n \quad \text{pointwise on } [-\tau, \infty).$$

By (3.7), we have

$$0 \leq f(t, x_n(t - T_1(t)), ..., x_n(t - T_m(t))) \leq f(t, \rho_1(t), ..., \rho_m(t))$$

for every $t \geq 0$ and all nonnegative integers n. As hypothesis (3.1) implies, in particular, that

$$\int_0^\infty f(t, \rho_1(t), ..., \rho_m(t))dt < \infty,$$

we can apply the Lebesgue dominated convergence theorem to obtain, for every $t \geq 0$,

$$\lim_{n\to\infty} \int_t^\infty f(s, x_n(s - T_1(s)), ..., x_n(s - T_m(s)))ds$$
$$= \int_t^\infty f(s, x(s - T_1(s)), ..., x(s - T_m(s)))ds.$$

Thus, we conclude that

$$\lim_{n\to\infty} (My_n)(t) = (My)(t) \quad \text{for every } t \geq 0.$$

Hence, we have

$$y(t) = \lim_{n\to\infty} y_{n+1}(t) = \lim_{n\to\infty} (My_n)(t) = (My)(t) \quad \text{for } t \geq 0$$

and consequently $y = My$, i.e., (1.5) holds. Also, (1.4) is satisfied. Therefore, by Proposition 1.1, (x, y) is a solution of the BVP (1.1)–(1.3). As $y \in Y$, (3.5) and (3.6) are satisfied. By (3.6), x is nonnegative on the interval $(0, \infty)$; hence, Lemma 2.1 guarantees that x is always positive on $(0, \infty)$ and, in addition, that y is necessarily positive on $[0, \infty)$. Thus, the solution (x, y) satisfies (3.3) and (3.4).

The proof of the theorem is complete.

It is evident that Lemma 2.1 plays a crucial role in proving Theorem 3.1. Moreover, one may easily see that the proof of Lemma 2.1 is essentially based on the use of the hypothesis that the initial function ϕ is positive on the interval $[-\tau, 0)$ (as well as on the assumptions (2.1) and (2.2)). This hypothesis is fundamental, because of the fact that $\tau > 0$, which is a consequence of the fact that the delays T_j $(j = 1, .., m)$ are positive on the interval $[0, \infty)$. It is clear that such a hypothesis cannot be posed in the case of the nonlinear two-dimensional ordinary differential systems, and hence Lemma 2.1 (and, consequently, Theorem 3.1) cannot be applied to the corresponding ordinary boundary value problem. More precisely, let us consider the nonlinear two-dimensional delay differential system

$$x'(t) = g(t, y(t)), \quad y'(t) = -f_0(t, x(t - \tau)), \tag{3.10}$$

where f_0 is a continuous real-valued function on $[0, \infty) \times \mathbb{R}$, and τ is a positive real constant. For $\tau = 0$, system (3.10) reduces to nonlinear two-dimensional ordinary differential system

$$x'(t) = g(t, y(t)), \quad y'(t) = -f_0(t, x(t)), \tag{3.11}$$

and the initial condition (1.2) becomes

$$x(0) = 0. \tag{3.12}$$

That is, when $\tau = 0$, the BVP (3.10), (1.2), (1.3) reduces to the BVP (3.11), (3.12), (1.3). Lemma 2.1 and Theorem 3.1 can be applied to the delay BVP (3.10), (1.2), (1.3), while these results are not applicable to the ordinary BVP (3.11), (3.12), (1.3).

4. APPLICATION TO SECOND ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

Consider the second order nonlinear delay differential equation

$$[r(t)x'(t)]' + f(t, x(t - T_1(t)), ..., x(t - T_m(t))) = 0, \tag{4.1}$$

where r *is a positive continuous real-valued function on the interval* $[0, \infty)$. We are interested in solutions of (4.1) on the *whole* interval $[0, \infty)$. By a *solution on* $[0, \infty)$ of (4.1), we mean a continuous real-valued function x defined on the interval $[-\tau, \infty)$, which is continuously differentiable on $[0, \infty)$ and such that rx' is continuously differentiable on $[0, \infty)$ and (4.1) is satisfied for all $t \geq 0$. With the delay differential equation (4.1), we associate the initial condition (1.2) as well as the condition

$$\lim_{t \to \infty} r(t)x'(t) = 0. \tag{4.2}$$

Equations (4.1), (1.2), (4.2) constitute a *boundary value problem* (*BVP*, for short) *on the half-line.* A *solution* of the BVP (4.1), (1.2), (4.2) is a solution on $[0, \infty)$ of (4.1) that satisfies the conditions (1.2) and (4.2).

The substitution $rx' = y$ transforms the second order nonlinear delay differential equation (4.1) into the equivalent nonlinear two-dimensional delay differential system

$$x'(t) = \frac{1}{r(t)} y(t), \qquad y'(t) = -f(t, x(t - T_1(t)), ..., x(t - T_m(t))). \tag{4.3}$$

By this substitution, the BVP (4.1), (1.2), (4.2) is transformed into the equivalent BVP (4.3), (1.2), (1.3), which is a special case of the BVP (1.1)–(1.3).

For our convenience, we introduce some notation. By R we will denote the *continuous real-valued function on the interval* $[0, \infty)$ defined by the formula

$$R(t) = \int_0^t \frac{ds}{r(s)} \quad \text{for } t \geq 0.$$

Clearly, $R(0) = 0$, and R *is positive on* $(0, \infty)$.

By specifying Theorem 3.1 to the BVP (4.3), (1.2), (1.3), we are led to the following result concerning the BVP (4.1), (1.2), (4.2).

Corollary 4.1. *Assume that the function f is positive on $[0, \infty) \times (0, \infty)^m$, i.e., (2.2) holds. Moreover, assume that, for each $t \geq 0$, the function $f(t, \cdot, ..., \cdot)$ is increasing on $[0, \infty)^m$.*

Let there exist a real number $c > 0$ so that

$$\int_0^\infty f(t, \sigma_1(t), ..., \sigma_m(t))\, dt \leq c,$$

where, for each $j \in \{1, ..., m\}$, the function σ_j depends on ϕ, c, r and is defined by

$$\sigma_j(t) = \begin{cases} \phi(t - T_j(t)), & \text{if } 0 \leq t \leq T_j(t) \\ cR(t - T_j(t)), & \text{if } t \geq T_j(t). \end{cases}$$

(Clearly, σ_j $(j = 1, ..., m)$ are nonnegative continuous real-valued functions on the interval $[0, \infty)$.) Then the BVP (4.1), (1.2), (4.2) has at least one solution x such that

$$0 < x(t) \leq cR(t) \quad \textit{for every } t > 0$$

and

$$0 < r(t)x'(t) \leq c \quad \textit{for every } t \geq 0.$$

By applying Corollary 4.1 to the particular case where $r(t) = 1$ for $t \geq 0$, we immediately arrive at the main result in the recent author's paper [31]. (Note that, in this particular case, we have $R(t) = t$ for $t \geq 0$.)

For the sake of completeness, we also give the application of Lemma 2.1 to the BVP (4.1), (1.2), (4.2). By specifying Lemma 2.1 to the BVP (4.3), (1.2), (1.3), we get the next result.

Assume that (2.2) *holds. Let x be a solution of the BVP* (4.1), (1.2), (4.2) *that is nonnegative on the interval* $(0, \infty)$. *Then x is always positive on* $(0, \infty)$; *moreover, x' is positive on* $[0, \infty)$ (*and so x is strictly increasing on* $[0, \infty)$).

In the particular case where $r(t) = 1$ for $t \geq 0$, the above result has been established by the author in [31].

Before closing this section, let us consider the particular case where the first equation of (1.1) is linear, i.e., the case of the nonlinear two-dimensional delay differential system

$$x'(t) = q(t)y(t), \qquad y'(t) = -f(t, x(t - T_1(t)), \dots, x(t - T_m(t))), \tag{4.4}$$

where q is a positive continuous real-valued function on the interval $[0, \infty)$. Theorem 3.1 can be applied to the BVP (4.4), (1.2), (1.3). On the other hand, we immediately see that (4.4) can be transformed into the equivalent second order nonlinear delay differential equation

$$\left[\frac{1}{q(t)}x'(t)\right]' + f(t, x(t - T_1(t)), \dots, x(t - T_m(t))) = 0. \tag{4.5}$$

With (4.5), we associate the initial condition (1.2) and the condition

$$\lim_{t\to\infty} \frac{1}{q(t)}x'(t) = 0. \tag{4.6}$$

It is remarkable that, instead of applying Theorem 3.1 to the BVP (4.4), (1.2), (1.3), one can apply Corollary 4.1 to the BVP (4.5), (1.2), (4.6).

5. APPLICATION TO NONLINEAR TWO-DIMENSIONAL DELAY DIFFERENTIAL SYSTEMS OF EMDEN-FOWLER TYPE. EXAMPLES

Consider the nonlinear two-dimensional delay differential system of Emden-Fowler type

$$x'(t) = q(t)\,|y(t)|^{\beta}\,\mathrm{sgn}y(t), \qquad y'(t) = -\sum_{j=1}^{m} p_j(t)\,|x(t - \tau_j)|^{\gamma_j}\,\mathrm{sgn}x(t - \tau_j), \tag{5.1}$$

where m *is a positive integer, q is a positive continuous real-valued function on the interval* $[0, \infty)$, p_j $(j = 1, \dots, m)$ *are nonnegative continuous real-valued functions* *on* $[0,\infty)$, τ_j $(j=1,...,m)$ *are positive real constants, and* β *and* γ_j $(j=1,...,m)$ *are positive real numbers. It will be supposed that*

$$\sum_{j=1}^{m} p_j(t) > 0 \quad \textit{for all } t \geq 0.$$

We notice that, as p_j $(j=1,...,m)$ are nonnegative on $[0,\infty)$, the last hypothesis means exactly that, *for each* $t \geq 0$, *there exists at least one index* $j \in \{1,...,m\}$ *so that* $p_j(t) > 0$.

Set

$$\tau = \max_{j=1,...,m} \tau_j.$$

(τ is a positive real number.) Our interest is concentrated on solutions of (5.1) on the *whole* interval $[0,\infty)$. A *solution on* $[0,\infty)$ of (5.1) is a pair of two continuous real-valued functions x and y defined on the intervals $[-\tau,\infty)$ and $[0,\infty)$, respectively, which are continuously differentiable on $[0,\infty)$ and satisfy (5.1) for all $t \geq 0$. The initial condition (1.2) as well as the condition (1.3) are associated with the delay differential system (5.1). Hence, we have the BVP (5.1), (1.2), (1.3).

For our convenience, we denote by Q the *continuous real-valued function on the interval* $[0,\infty)$ defined by the formula

$$Q(t) = \int_0^t q(s)ds \quad \text{for } t \geq 0.$$

Note that $Q(0) = 0$ and that Q *is positive on* $(0,\infty)$.

By applying Theorem 3.1 to the particular case of the BVP (5.1), (1.2), (1.3), we are led to the following corollary.

Corollary 5.1. *Let there exist a real number* $c > 0$ *so that*

$$\sum_{j=1}^{m} \int_0^{\tau_j} [\phi(t-\tau_j)]^{\gamma_j} p_j(t)dt + \sum_{j=1}^{m} c^{\beta\gamma_j} \int_{\tau_j}^{\infty} [Q(t-\tau_j)]^{\gamma_j} p_j(t)dt \leq c.$$

Then the BVP (5.1), (1.2), (1.3) *has at least one solution* (x,y) *such that*

$$0 < x(t) \leq c^{\beta} Q(t) \quad \textit{for every } t > 0 \tag{5.2}$$

and (3.4) *holds.*

Now, in order to present some examples demonstrating the applicability of our theorem, we shall concentrate on nonlinear two-dimensional Emden-Fowler type delay differential systems with one constant delay.

Let us consider the delay differential system of Emden-Fowler type

$$x'(t) = q(t)\,|y(t)|^{\beta}\,\text{sgn}y(t), \qquad y'(t) = -p(t)\,|x(t-\tau)|^{\gamma}\,\text{sgn}x(t-\tau), \tag{5.3}$$

where p *and* q *are positive continuous real-valued functions on the interval* $[0,\infty)$, τ *is a positive real constant, and* β *and* γ *are positive real numbers.*

In the particular case of the BVP (5.3), (1.2), (1.3), Corollary 5.1 is formulated as follows.

Let there exist a real number $c > 0$ *so that*

$$\int_0^{\tau} [\phi(t-\tau)]^{\gamma} p(t)dt + c^{\beta\gamma} \int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)dt \leq c. \tag{5.4}$$

Then the BVP (5.3), (1.2), (1.3) *has at least one solution* (x, y) *such that* (5.2) *and* (3.4) *hold.*

Example 5.2. Consider the BVP (5.3), (1.2), (1.3) with $\beta\gamma = 1$. In this case, condition (5.4) is written as

$$\int_0^T [\phi(t-\tau)]^\gamma p(t)dt + c\int_T^\infty [Q(t-\tau)]^\gamma p(t)dt \leq c$$

or

$$\int_0^T [\phi(t-\tau)]^\gamma p(t)dt \leq c\left\{1 - \int_T^\infty [Q(t-\tau)]^\gamma p(t)dt\right\}. \tag{5.5}$$

We see that, if

$$\int_T^\infty [Q(t-\tau)]^\gamma p(t)dt < 1, \tag{5.6}$$

then inequality (5.5) holds true (as an equality) for

$$c = \frac{\int_0^T [\phi(t-\tau)]^\gamma p(t)dt}{1 - \int_T^\infty [Q(t-\tau)]^\gamma p(t)dt}. \tag{5.7}$$

Clearly, c is a positive real number. So, we arrive at the next result.

Assume that $\beta\gamma = 1$. *Let condition* (5.6) *be satisfied, and let* $c > 0$ *be the real number given by* (5.7). *Then the BVP* (5.3), (1.2), (1.3) *has at least one solution* (x, y) *such that* (5.2) *and* (3.4) *hold.*

Example 5.3. Let us consider the BVP (5.3), (1.2), (1.3) with $\beta\gamma = \frac{1}{2}$. Here, condition (5.4) becomes

$$\int_0^T [\phi(t-\tau)]^\gamma p(t)dt + c^{1/2}\int_T^\infty [Q(t-\tau)]^\gamma p(t)dt \leq c,$$

namely

$$c - \left\{\int_T^\infty [Q(t-\tau)]^\gamma p(t)dt\right\} c^{1/2} - \int_0^T [\phi(t-\tau)]^\gamma p(t)dt \geq 0. \tag{5.8}$$

Assume that

$$\int_T^\infty [Q(t-\tau)]^\gamma p(t)dt < \infty. \tag{5.9}$$

Following the lines of Example 1 in the author's paper [31], we can show that (5.8) holds with $c > 0$ if and only if

$$c \geq \left(\frac{1}{2}\int_T^\infty [Q(t-\tau)]^\gamma p(t)dt + \sqrt{\left\{\frac{1}{2}\int_T^\infty [Q(t-\tau)]^\gamma p(t)dt\right\}^2 + \int_0^T [\phi(t-\tau)]^\gamma p(t)dt}\right)^2.$$

Thus, we conclude that (5.8) is valid (as an equality) for

$$c = \left(\frac{1}{2} \int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)dt + \sqrt{\left\{ \frac{1}{2} \int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)dt \right\}^2 + \int_0^{\tau} [\phi(t-\tau)]^{\gamma} p(t)dt} \right)^2 . \tag{5.10}$$

Hence, we obtain the following result.

Assume that $\beta\gamma = \frac{1}{2}$. *Let condition* (5.9) *be satisfied, and let* $c > 0$ *be the real number given by* (5.10). *Then the BVP* (5.3), (1.2), (1.3) *has at least one solution* (x, y) *such that* (5.2) *and* (3.4) *hold.*

Example 5.4. Consider the case of the BVP (5.3), (1.2), (1.3) with $\beta\gamma = 2$. In this case, condition (5.4) takes the form

$$\int_0^{\tau} [\phi(t-\tau)]^{\gamma} p(t)dt + c^2 \int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)dt \leq c,$$

i.e.,

$$\left\{ \int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)dt \right\} c^2 - c + \int_0^{\tau} [\phi(t-\tau)]^{\gamma} p(t)dt \leq 0. \tag{5.11}$$

After a long analysis similar to that used by the author in Example 2 in [31], we can conclude that, if

$$\left\{ \int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)dt \right\} \int_0^{\tau} [\phi(t-\tau)]^{\gamma} p(t)dt \leq \frac{1}{4}, \tag{5.12}$$

then (5.11) holds (as an equality) for

$$c = \frac{1 - \sqrt{1 - 4 \left\{ \int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)dt \right\} \int_0^{\tau} [\phi(t-\tau)]^{\gamma} p(t)dt}}{2 \int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)dt} . \tag{5.13}$$

Thus, we are led to the next result.

Assume that $\beta\gamma = 2$. *Let condition* (5.12) *be satisfied, and let* $c > 0$ *be the real number given by* (5.13). *Then the BVP* (5.3), (1.2), (1.3) *has at least one solution* (x, y) *such that* (5.2) *and* (3.4) *hold.*

Before closing this section and ending the paper, we note that, by the use of the particular results obtained in the above general examples, one can construct specific examples in which our theorem is applicable. For such specific examples to the special case of second order nonlinear delay differential equations, we refer to the recent our paper [31].

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A SURVEY ON THE OSCILLATION OF DELAY AND DIFFERENCE EQUATIONS WITH VARIABLE DELAY

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ABSTRACT

Consider the first-order linear delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

where $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$, $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$, and the (discrete analogue) difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, ..., \tag{1$'$}$$

where $\Delta x(n) = x(n+1) - x(n)$, $p(n)$ is a sequence of nonnegative real numbers and $\tau(n)$ is a nondecreasing sequence of integers such that $\tau(n) \leq n - 1$ for all $n \geq 0$ and $\lim_{n\to\infty} \tau(n) = \infty$. Optimal conditions for the oscillation of all solutions to the above equations are presented.

1 Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions to the differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

where the functions $p, \tau \in C([t_0,\infty), \mathbb{R}^+)$ (here $\mathbb{R}^+ = [0,\infty)$), $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$, has been the subject of many investigations. See, for example, [11, 15, 17, 21–26, 28, 29–32, 33-42, 44, 47-52, 54, 55, 59, 60, 66, 73–80, 82-84, 90] and the references cited therein.

By a solution of Eq.(1) we understand a continuously differentiable function defined on $[\tau(T_0), \infty)$ for some $T_0 \geq t_0$ and such that Eq.(1) is satisfied for $t \geq T_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*.

The oscillation theory of the (discrete analogue) delay difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, ..., \tag{1$'$}$$

where $\Delta x(n) = x(n+1) - x(n)$, $p(n)$ is a sequence of nonnegative real numbers and $\tau(n)$ is a nondecreasing sequence of integers such that $\tau(n) \leq n-1$ for all $n \geq 0$ and $\lim_{n\to\infty} \tau(n) = \infty$, has also attracted growing attention in the last decades, especially in the case where the delay $n - \tau(n)$ is a constant, that is, in the special case of the difference equation,

$$\Delta x(n) + p(n)x(n-k) = 0, \quad n = 0, 1, 2, ... \tag{1$''$}$$

where k is a positive integer. The reader is referred to [5–10, 12, 13, 16, 18-20, 43, 46, 53, 56, 57, 61, 62, 63-65, 67-72, 81, 85-89] and the references cited therein.

By a solution of Eq.(1)$'$ we mean a sequence $x(n)$ which is defined for $n \geq -k$ and which satisfies (1)$'$ for $n \geq 0$. A solution $x(n)$ of Eq.(1)$'$ is said to be *oscillatory* if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative, and otherwise the solution is said to be *nonoscillatory*. (Analogously for Eq.(1)$''$.)

In this paper our main purpose is to present the state of the art on the oscillation of all solutions to Eq.(1) especially in the case where

$$0 < \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e} \quad \text{and} \quad \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds < 1,$$

and (the discrete analogues) for Eq.(1)$'$ when

$$\liminf_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) \leq \frac{1}{e} \quad \text{and} \quad \limsup_{n\to\infty} \sum_{i=\tau(n)}^{n} p(i) < 1.$$

2 Oscillation Criteria for Eq. (1)

In this section we study the delay equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0. \tag{1}$$

where the functions $p, \tau \in C([t_0,\infty), \mathbb{R}^+)$, $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

The first systematic study for the oscillation of all solutions to Eq.(1) was made by Myshkis. In 1950 [58] he proved that every solution of Eq.(1) oscillates if

$$\limsup_{t\to\infty}[t - \tau(t)] < \infty \quad \text{and} \quad \liminf_{t\to\infty}[\mathrm{t} - \tau(\mathrm{t})] \liminf_{t\to\infty} \mathrm{p(t)} > \frac{1}{\mathrm{e}}. \tag{C_1}$$

In 1972, Ladas, Lakshmikantham and Papadakis [44] proved that the same conclusion holds if

$$A := \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1. \tag{C_2}$$

In 1979, Ladas [42] established integral conditions for the oscillation of Eq.(1) with constant delay. Tomaras [77-79] extended this result to Eq.(1) with variable delay. For related results see Ladde [49-51]. The following most general result is due to Koplatadze and Canturija [37].

If

$$\mathfrak{a} := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}, \tag{C_3}$$

then all solutions of Eq.(1) oscillate; If

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds < \frac{1}{e}, \tag{N_1}$$

then Eq.(1) has a nonoscillatory solution.

It is obvious that there is a gap between the conditions (C_2) and (C_3) when the limit $\lim\limits_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds$ does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors.

In 1988, Erbe and Zhang [26] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible nonoscillatory solutions $x(t)$ of Eq.(1). Their result says that all the solutions of Eq.(1) are oscillatory, if $0 < \mathfrak{a} \le \frac{1}{e}$ and

$$A > 1 - \frac{\mathfrak{a}^2}{4}. \qquad (C_4)$$

Since then several authors tried to obtain better results by improving the upper bound for $x(\tau(t))/x(t)$.

In 1991, Jian [35] derived the condition

$$A > 1 - \frac{\mathfrak{a}^2}{2(1-\mathfrak{a})}, \qquad (C_5)$$

while in 1992, Yu and Wang [83] and Yu, Wang, Zhang and Qian [84] obtained the condition

$$A > 1 - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2}. \qquad (C_6)$$

In 1990, Elbert and Stavroulakis [23] and in 1991 Kwong [41], using different techniques, improved (C_4), in the case where $0 < \mathfrak{a} \le \frac{1}{e}$, to the conditions

$$A > 1 - (1 - \frac{1}{\sqrt{\lambda_1}})^2 \qquad (C_7)$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1}, \qquad (C_8)$$

respectively, where λ_1 is the smaller real root of the equation $\lambda = e^{\mathfrak{a}\lambda}$.

In 1994, Koplatadze and Kvinikadze [38] improved (C_6), while in 1998, Philos and Sficas [59] and in 1999, Zhou and Yu [90] and Jaroš and Stavroulakis [34] derived the conditions

$$A > 1 - \frac{\mathfrak{a}^2}{2(1-\mathfrak{a})} - \frac{\mathfrak{a}^2}{2}\lambda_1, \qquad (C_9)$$

$$A > 1 - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2} - (1 - \frac{1}{\sqrt{\lambda_1}})^2, \qquad (C_{10})$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2}, \qquad (C_{11})$$

respectively.

Consider Eq.(1) and assume that $\tau(t)$ is continuously differentiable and that there exists $\theta > 0$ such that $p(\tau(t))\tau'(t) \geq \theta p(t)$ eventually for all t. Under this additional condition, in 2000, Kon, Sficas and Stavroulakis [36] and in 2003, Sficas and Stavroulakis [60] established the conditions

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \mathfrak{a} - \sqrt{(1 - \mathfrak{a})^2 - 4\Theta}}{2} \qquad (2.1)$$

and

$$A > \frac{\ln \lambda_1}{\lambda_1} - \frac{1 + \sqrt{1 + 2\theta - 2\theta\lambda_1 M}}{\theta\lambda_1} \qquad (2.2)$$

respectively, where $\Theta = \frac{e^{\lambda_1 \theta \mathrm{a}} - \lambda_1 \theta \mathrm{a} - 1}{(\lambda_1 \theta)^2}$ and $M = \frac{1 - \mathrm{a} - \sqrt{(1-\mathrm{a})^2 - 4\Theta}}{2}$.

Remark 2.1. ([36], [60]) Observe that when $\theta = 1$, then $\Theta = \frac{\lambda_1 - \lambda_1 \mathrm{a} - 1}{{\lambda_1}^2}$, and (2.1) reduces to

$$A > 2\mathfrak{a} + \frac{2}{\lambda_1} - 1, \qquad (C_{12})$$

while in this case it follows that $M = 1 - \mathfrak{a} - \frac{1}{\lambda_1}$ and (2.2) reduces to

$$A > \frac{\ln \lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2\mathfrak{a}\lambda_1}}{\lambda_1}. \qquad (C_{13})$$

In the case where $\mathfrak{a} = \frac{1}{e}$, then $\lambda_1 = e$, and (C_{13}) leads to

$$A > \frac{\sqrt{7 - 2e}}{e} \approx 0.459987065.$$

It is to be noted that as $\mathfrak{a} \to 0$, then all the previous conditions (C_4) – (C_{12}) reduce to the condition (C_2), i.e. $A > 1$. However, the condition (C_{13}) leads to

$$A > \sqrt{3} - 1 \approx 0.732,$$

which is an essential improvement. Moreover (C_{13}) improves all the above conditions when $0 < \mathfrak{a} \leq \frac{1}{e}$ as well. Note that the value of the lower bound on A can not be less than $\frac{1}{e} \approx 0.367879441$. Thus the aim is to establish a condition which leads to a value *as close as possible to* $\frac{1}{e}$. For illustrative purpose, we give the values of the lower bound on A under these conditions when $\alpha = \frac{1}{e}$.

(C_4):	0.966166179
(C_5):	0.892951367
(C_6):	0.863457014
(C_7):	0.845181878
(C_8):	0.735758882
(C_9):	0.709011646
(C_{10}):	0.708638892
(C_{11}):	0.599215896
(C_{12}):	0.471517764
(C_{13}):	0.459987065

We see that the condition (C_{13}) essentially improves all the known results in the literature.

Example 2.1 ([60]) Consider the delay differential equation

$$x'(t) + px(t - q\sin^2\sqrt{t} - \frac{1}{pe}) = 0,$$

where $p > 0$, $q > 0$ and $pq = 0.46 - \frac{1}{e}$. Then

$$\alpha = \liminf_{t\to\infty} \int_{\tau(t)}^{t} pds = \liminf_{t\to\infty} p(q\sin^2\sqrt{t} + \frac{1}{pe}) = \frac{1}{e}$$

and

$$A = \limsup_{t\to\infty} \int_{\tau(t)}^{t} pds = \limsup_{t\to\infty} p(q\sin^2\sqrt{t} + \frac{1}{pe}) = pq + \frac{1}{e} = 0.46.$$

Thus, according to Remark 2.1, all solutions of this equation oscillate. Observe that none of the conditions (C_4)-(C_{12}) apply to this equation.

Following this historical (and chronological) review we also mention that in the case where

$$\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \frac{1}{e}$$

this problem has been studied in 1995, by Elbert and Stavroulakis [24], by Kozakiewicz [39], Li [54, 55] and in 1996, by Domshlak and Stavroulakis [22].

3 Oscillation Criteria for Eq. (1)$'$

In this section we study the difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, ..., \tag{1'}$$

where $\Delta x(n) = x(n+1) - x(n)$, $p(n)$ is a sequence of nonnegative real numbers and $\tau(n)$ is a nondecreasing sequence of integers such that $\tau(n) \leq n-1$ for all $n \geq 0$ and $\lim_{n\to\infty} \tau(n) = \infty$.

In the special case where the delay $n - \tau(n)$ is a constant, the delay difference equation (1)$'$ becomes

$$\Delta x(n) + p(n)x(n-k) = 0, \quad n = 0, 1, 2, ... \tag{1''}$$

where k is a positive integer.

In 1981, Domshlak [12]was the first who studied this problem in the case where $k = 1$.Then, in 1989, Erbe and Zhang [27] established that all solutions of Eq.(1)$''$ are oscillatory if

$$\liminf_{n\to\infty} p(n) > \frac{k^k}{(k+1)^{k+1}} \tag{3.1}$$

or

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p(i) > 1. \tag{C_2''}$$

In the same year, 1989, Ladas, Philos and Sficas [46] proved that a sufficient condition for all solutions of Eq.(1)$''$ to be oscillatory is that

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > \left(\frac{k}{k+1}\right)^{k+1} \tag{C_3''}$$

Therefore they improved the condition (3.1) by replacing the $p(n)$ of (3.1) by the arithmetic mean of $p(n-k), ..., p(n-1)$ in $(C_3)''$.

Concerning the constant $\frac{k^k}{(k+1)^{k+1}}$ in (3.1) it should be emphasized that, as it is shown in [27], if

$$\sup p(n) < \frac{k^k}{(k+1)^{k+1}}$$

then Eq.(1)″ has a nonoscillatory solution.

In 1990, Ladas [43] conjectured that Eq.(1)″ has a nonoscillatory solution if

$$\sum_{i=n-k}^{n-1} p(i) < \left(\frac{k}{k+1}\right)^{k+1}$$

holds eventually. However, a counterexample to this conjecture was given in 1994, by Yu, Zhang and Wang [86].

It is interesting to establish sufficient oscillation conditions for the equation (1)″ in the case where neither $(C_2)''$ nor $(C_3)''$ is satisfied.

In 1995, the following oscillation criterion was established by Stavroulakis [63]:

If $0 < \alpha_0 \leq \left(\frac{k}{k+1}\right)^{k+1}$, where

$$\alpha_0 = \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p(i)$$

then the condition

$$\limsup_{n\to\infty} p(n) > 1 - \frac{\alpha_0^2}{4} \tag{3.2}$$

implies that all solutions of Eq.(1)″ oscillate. In 2004, the same author [64] improved the condition (3.2) to the following

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\alpha_0^2}{4} \tag{C_4}''$$

or

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \alpha_0^k, \tag{3.3}$$

while in 2006, Chatzarakis and Stavroulakis [5], established the condition

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\alpha_0^2}{2(2-\alpha_0)}. \tag{3.4}$$

Also, Chen and Yu [6] obtained the following oscillation condition

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p(i) > 1 - \frac{1-\alpha_0 - \sqrt{1-2\alpha_0-\alpha_0^2}}{2}. \tag{$(C_6)''$}$$

Remark 3.1 Observe that the conditions $(C_2)''$, $(C_3)''$, $(C_4)''$ and $(C_6)''$ are the discrete analogues of the conditions (C_2), (C_3), (C_4) and (C_6) respectively for Eq.$(1)''$.

In the case of Eq.$(1)'$ with a general delay argument $\tau(n)$, from Chatzarakis, Koplatadze and Stavroulakis [2], it follows the following

Theorem 3.1 ([2]) *If*

$$\limsup_{n\to\infty} \sum_{i=\tau(n)}^{n} p(i) > 1 \tag{$(C_2)'$}$$

then all solutions of Eq. $(1)'$ *oscillate.*

This result generalizes the oscillation criterion $(C_2)''$. Also Chatzarakis, Koplatadze and Stavroulakis [3] extended the oscillation criterion $(C_3)''$ to the general case of Eq. $(1)'$. More precisely, the following theorem has been established in [3].

Theorem 3.2 ([3]) *Assume that*

$$\limsup_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) < +\infty \tag{3.5}$$

and

$$\alpha := \liminf_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) > \frac{1}{e}. \tag{$(C_3)'$}$$

Then all solutions of Eq.$(1)'$ *oscillate.*

Remark 3.2 It is to be pointed out that the conditions $(C_2)'$ and $(C_3)'$ are the discrete analogues of the conditions (C_2) and (C_3) and also the analogues of the conditions $(C_2)''$ and $(C_3)''$ for Eq.$(1)'$ in the case of a general delay argument $\tau(n)$.

Remark 3.3 ([3]). The condition $(C_3)'$ is optimal for Eq.(1)$'$ under the assumption that $\lim_{n\to+\infty}(n-\tau(n))=\infty$, since in this case the set of natural numbers increases infinitely in the interval $[\tau(n), n-1]$ for $n\to\infty$.

Now, we are going to present an example to show that the condition $(C_3)'$ is optimal, in the sense that it cannot be replaced by the non-strong inequality.

Example 3.1 ([3]) Consider Eq.(1)$'$, where

$$\tau(n)=[\beta n],\ p(n)=\left(n^{-\lambda}-(n+1)^{-\lambda}\right)([\beta n])^{\lambda},\ \beta\in(0,1),\ \lambda=-\ln^{-1}\beta \tag{3.6}$$

and $[\beta n]$ denotes the integer part of βn.

It is obvious that

$$n^{1+\lambda}\left(n^{-\lambda}-(n+1)^{-\lambda}\right)\to\lambda \quad \textit{for}\ \ n\to\infty.$$

Therefore

$$n\left(n^{-\lambda}-(n+1)^{-\lambda}\right)([\beta n])^{\lambda}\longrightarrow\frac{\lambda}{e}\quad \textit{for}\ \ n\to\infty. \tag{3.7}$$

Hence, in view of (3.6) and (3.7), we have

$$\begin{aligned}\liminf_{n\to\infty}\sum_{i=\tau(n)}^{n-1}p(i) &= \frac{\lambda}{e}\ \liminf_{n\to\infty}\sum_{i=[\beta n]}^{n-1}\frac{e}{\lambda}i\left(i^{-\lambda}-(i+1)^{-\lambda}\right)([\beta i])^{\lambda}.\frac{1}{i}\\ &= \frac{\lambda}{e}\ \liminf_{n\to\infty}\sum_{i=[\beta n]}^{n-1}\frac{1}{i}=\frac{\lambda}{e}\ \ln\frac{1}{\beta}=\frac{1}{e}\end{aligned}$$

or

$$\liminf_{n\to\infty}\sum_{i=\tau(n)}^{n-1}p(i)=\frac{1}{e}. \tag{3.8}$$

Observe that all the conditions of Theorem 3.2 are satisfied except the condition $(C_3)'$. In this case it is not guaranteed that all solutions of Eq.(1)$'$ oscillate. Indeed, it is easy to see that the function $u=n^{-\lambda}$ is a positive solution of Eq.(1)$'$.

As it has been mentioned above, it is an interesting problem to find new sufficient conditions for the oscillation of all solutions of the delay difference equation $(1)'$, in the case where neither $(C_2)'$ nor $(C_3)'$ is satisfied.

In 2008, Chatzarakis, Koplatadze and Stavroulakis [2] investigated for the first time this question for the difference equation $(1)'$ in the case of a general delay argument $\tau(n)$ and derived the following theorem.

Theorem 3.3 ([2]) *Assume that* $0 < \alpha \leq \frac{1}{e}$. *Then we have:*

(I) *If*

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \left(1 - \sqrt{1-\alpha}\right)^2 \tag{3.9}$$

then all solutions of Eq.$(1)'$ *oscillate.*

(II) *If in addition,*

$$p(n) \geq 1 - \sqrt{1-\alpha} \text{ for all large } n, \tag{3.10}$$

and

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \alpha \frac{1-\sqrt{1-\alpha}}{\sqrt{1-\alpha}} \tag{3.11}$$

then all solutions of Eq.$(1)'$ *oscillate.*

Recently, the above result was improved in [4] as follows.

Theorem 3.4 ([4]) (I) *If* $0 < \alpha \leq \frac{1}{e}$ *and*

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{2}\left(1 - \alpha - \sqrt{1-2\alpha}\right) \tag{3.12}$$

then all solutions of Eq.$(1)'$ *oscillate.*

(II) *If* $0 < \alpha \leq 6 - 4\sqrt{2}$ *and in addition,*

$$p(n) \geq \frac{\alpha}{2} \text{ for all large } n, \tag{3.13}$$

and

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{4}\left(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}\right) \tag{3.14}$$

then all solutions of Eq.$(1)'$ *are oscillatory.*

Remark 3.4 Observe the following:
(i) When $0 < \alpha \leq \frac{1}{e}$, it is easy to verify that

$$\frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha}\right) > \left(1 - \sqrt{1 - \alpha}\right)^2,$$

and therefore the inequality (3.12) improves the inequality (3.9).

(ii) When $0 < \alpha \leq 6 - 4\sqrt{2}$, because

$$1 - \sqrt{1 - \alpha} > \frac{\alpha}{2},$$

we see that the assumption (3.13) is weaker than the assumption (3.10), and moreover, we can show that

$$\frac{1}{4}\left(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}\right) > \alpha\frac{1 - \sqrt{1 - \alpha}}{\sqrt{1 - \alpha}}$$

and so the inequality (3.14) is an improvement of the inequality (3.11).

(iii) When $0 < \alpha \leq \frac{1}{e}$, it is easy to see that

$$\frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}\right) > \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha}\right)$$

and therefore, in the case of Eq.(1)$''$, the condition $(C_6)''$ is weaker than the condition (3.12).

Observe, however, that when $0 < \alpha \leq 6 - 4\sqrt{2}$, it is easy to show that

$$\frac{1}{4}\left(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}\right) > \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}\right),$$

and therefore in this case and when (3.13) holds, inequality (3.14) improves the inequality $(C_6)''$ and especially, when $\alpha = 6 - 4\sqrt{2} \simeq 0.3431457$, the lower bound in $(C_6)''$ is 0.8929094 while in (3.14) is 0.7573593.

We illustrate by the following example.

Example 3.2 ([4]) Consider the equation

$$\Delta x(n) + p(n)x(n - 2) = 0,$$

where

$$p(3n) = \frac{1474}{10000}, \quad p(3n+1) = \frac{1488}{10000}, \quad p(3n+2) = \frac{6715}{10000}, \qquad n = 0, 1, 2, ...$$

Here $k = 2$ and it is easy to see that

$$\alpha_0 = \liminf_{n\to\infty} \sum_{j=n-2}^{n-1} p(j) = \frac{1474}{10000} + \frac{1488}{10000} = 0.2962 < \left(\frac{2}{3}\right)^3 \simeq 0.2962963,$$

and

$$\limsup_{n\to\infty} \sum_{j=n-2}^{n} p(j) = \frac{1474}{10000} + \frac{1488}{10000} + \frac{6715}{10000} = 0.9677.$$

Observe that

$$0.9677 > 1 - \frac{1}{2}\left(1 - \alpha_0 - \sqrt{1 - 2\alpha_0}\right) \simeq 0.967317794,$$

that is, condition (3.12) of Theorem 3.4 is satisfied and therefore all solutions oscillate. Also, condition $(C_6)''$ is satisfied. Observe, however, that

$$0.9677 < 1,$$

$$\alpha_0 = 0.2962 < \left(\frac{2}{3}\right)^3 \simeq 0.2962963,$$

$$0.9677 < 1 - \left(1 - \sqrt{1 - \alpha_0}\right)^2 \simeq 0.974055774,$$

and therefore none of the conditions $(C_2)''$, $(C_3)''$ and (3.9) is satisfied.

If, on the other hand, in the above equation

$$p(3n) = p(3n+1) = \frac{1481}{10000}, \quad p(3n+2) = \frac{6138}{10000}, \qquad n = 0, 1, 2, ...,$$

it is easy to see that

$$\alpha_0 = \liminf_{n\to\infty} \sum_{j=n-2}^{n-1} p(j) = \frac{1481}{10000} + \frac{1481}{10000} = 0.2962 < \left(\frac{2}{3}\right)^3 \simeq 0.2962963,$$

and

$$\limsup_{n\to\infty} \sum_{j=n-2}^{n} p(j) = \frac{1481}{10000} + \frac{1481}{10000} + \frac{6138}{10000} = 0.91.$$

Furthermore, it is clear that

$$p(n) \geq \frac{\alpha_0}{2} \quad \text{for all large } n.$$

In this case

$$0.91 > 1 - \frac{1}{4}\left(2 - 3\alpha_0 - \sqrt{4 - 12\alpha_0 + \alpha_0^2}\right) \simeq 0.904724375,$$

that is, condition (3.14) of Theorem 3.4 is satisfied and therefore all solutions oscillate. Observe, however, that

$$0.91 < 1,$$

$$\alpha_0 = 0.2962 < \left(\frac{2}{3}\right)^3 \simeq 0.2962963,$$

$$0.91 < 1 - \left(1 - \sqrt{1 - \alpha_0}\right)^2 \simeq 0.974055774,$$

$$0.91 < 1 - \frac{1}{2}\left(1 - \alpha_0 - \sqrt{1 - 2\alpha_0 - \alpha_0^2}\right) \simeq 0.930883291,$$

and therefore none of the conditions $(C_2)''$, $(C_3)''$, (3.9) and $(C_6)''$ is satisfied.

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